Differential geometry of generalized almost quaterniohic structures, II

V.F. Kirichenko, O.E. Arseneva

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Abstract

The generalized conseption of self-dual and anti-self dual forms and manifolds is built. The complete classification of important classes of self-dual and anti-self-dual generalized Kaehler manifolds is obtained.

Contents

The present paper is the immediate continuation of Part 1 (see page 00-00). Herein we shall refer to the results of Paper 1. The main purpose of the present paper is studying geometry originality of 4-dimensional (pseudo-) Riemannian manifolds. This originality is contained in the existence of a self-dual structure on such manifolds. Then we investigate generalization possibilities of the structure on the manifolds of greater dimensions. Such investigation is created on the base of the concept of generalized almost quaternionic structures which was developed in the previous paper.

Section 4 introduces and investigates the class of vertical type AQ_α-structures generalizing the class of quaternionic-Kaehler structures, and finds the criterion of the structural bundle of the structure being Einsteinian that generalized the well-known Atiyah-Hitchin-Singer criterion of 4-dimensional oriented Riemannian manifolds being Einsteinian in terms of self-dual forms on the manifolds. We also prove the theorem of generalized quaternionic-Kaehler manifolds of dimension greater than 4 being Einsteinian that generalizes the well-known result of M.Berger.

Section 5 introduces and investigates the notion of t-conformal-semiflat AQ_α-manifold generalizing the classical notion of self-dual and anti-self-dual 4-dimensional Riemannian manifolds in the case AQ_α-manifolds ofarbitrary dimension. We introduce the notion of twistor curvature tensor of a vertical type AQ_α-manifold and prove that the twistor curvature tensor of t-conformal-semiflat AQ_α-manifold is an algebraic curvature tensor. We also prove that anti-self-dual AQ_α-manifolds as well as generalized quaternionic-Kaehler manifolds are the manifolds of pointwise constant twistor curvature. It is proved that generalized hyper-Kaehler manifolds are zero twistor curvature manifolds, and they also are zero Ricci curvature manifolds. The result generalizes the well-known result of M.Berger.

In Section 6 we prove that a 4-dimensional generalized Kaehler manifold is self-dual iff its (generalized) Bochner curvature tensor is equal to zero. Moreover, we get a complete classification of 4-dimensional self-dual non-exceptional manifolds of constant scalar curvature. It is proved that a 4-dimensional generalized Kaehler manifold is anti-self-dual if and only if its scalar curvature is equal to zero. It is also proved that a 4-dimensional compact regular spinor manifold carrying a classical type Kaehler structure is anti-self-dual if and only if it is Ricci-flat. The above results essentially generalize the well-known results of N. Hitchin, J.-P. Bourguignon, A. Derdzinski, B.-Y. Chen and M. Itoh.

4 Generalized Almost Quaternionic Manifolds of Vertical Type

The study of geometry of 4-dimensional (pseudo-) Riemannian manifolds is of great importance in the modern geometric investigations since such manifolds

are significant in mathematical physics. It was the features of the geometry that gave rise to the wide use of the well-known Penrose Twistor Programme in modern investigations in the theory of gravitation and Yang-Mills fields. These features are determined to a great extent by the existence of a natural almost quaternionic structure on 4-dimensional (pseudo-) Riemannian manifolds. Naturally the question arises: what geometrical features of 4-dimensional manifolds admit generalization onto almost quaternionic manifolds of arbitrary dimension ? For example, it is known that with every almost quaternionic manifold there is naturally associated a twistor bundle, and a number of important results of Penrose twistor geometry are naturally, though non-trivially, generalized on other types of almost quaternionic manifolds, for example, on quaternionic-Kaehler ones [1],[2].

The present chapter distinguishes the class of generalized almost quaternionic structures for which similar extrapolation looks most natural. They are the so-called *generalizeg almost quaternionic structures of vertical type* that, as we shall see below, generalize quaternionic-Kaehler structures. The main result of the chapter shows that structural bundle of the manifold carrying the AQ_α-structure of vertical type is Einsteinian if and only if its Kaehler module is invariant with respect to Riemann-Christoffel endomorphism. This widely generalizes the classical Atiyah-Hitchin-Singer result giving a criterion of 4-dimensional Riemannian manifolds being Einsteinian in terms of self-dual forms [3]. It is also proved that generalized quaternionic-Kaehler manifold of dimension greater than 4 is an Einsteinian manifold, that generalizes the well-known Berger theorem of quaternionic-Kaehler manifolds being Einsteinian [1].

4.1 Self-dual and anti-self-dual forms on AQ_α-manifolds

Let M be an AQ_α-manifold, Q be its structural bundle. Note that algebra $\mathbf{H}_\alpha$ of generalized quaternions admits a canonical representation (of the first kind) $\lambda : \mathbf{H}_\alpha \to \operatorname{End}\mathbf{H}_\alpha,\ \lambda(q)(X) = qX;\ X \in \mathbf{H}_\alpha$. Indeed, $\lambda(q_1\,q_2)(X) = (q_1\,q_2)X = q_1(q_2\,X) = \lambda(q_1)\circ\lambda(q_2)(X)$, i.e. $\lambda(q_1\,q_2) = \lambda(q_1)\circ\lambda(q_2);\quad q_1, q_2 \in \mathbf{H}_\alpha$. Let $\{\,1, j_1, j_2, j_3\,\}$ be orthonormalized basis of the space $\mathbf{H}_\alpha, j_3 = j_1 j_2, q = a + bj_1 + cj_2 + dj_3 \in \mathbf{H}_\alpha$. Then in the basis

$$(\lambda(q)^i{}_j) = \begin{pmatrix} a & \alpha b & \alpha c & -d \\ b & a & \alpha d & -\alpha c \\ c & -\alpha d & a & \alpha b \\ d & -c & b & a \end{pmatrix}.$$

Thus, any α-quaternion admits identification with endomorphism of 4-dimensional real space $\mathbf{H}_\alpha$. Lowering index of the endomorphism we get a tensor of the type (2,0) on this linear space that will be skew-symmetric iff $a = 0$, i.e. α-quaternion q is purely imaginary. In its turn, fibre bundle Q has a natural metric $g = (\cdot,\cdot)$ generated by the metric of algebra $\mathbf{H}_\alpha$. Using it we lower index of the purely imaginary α-quaternion regarded in it canonical representation

and get a 2-form ω on the space of fibre bundle Q, that we call *a fundamental form of the α-quaternion.*

Similarly, we can consider canonical representation of the second kind $\mu : \mathbf{H}_\alpha \to \operatorname{End}\mathbf{H}_a$; $\mu(q)(X) = Xq$; $X \in \mathbf{H}_\alpha$. In this case $\mu(q_1\,q_2) = \mu(q_2)\circ\mu(q_1)$; $q_1, q_2 \in \mathbf{H}_\alpha$. Here the mentioned α-quaternion q in orthonormalized basis $\{\,1, j_1, j_2, j_3\,\}$, $j_3 = j_1 j_2$, will be represented by the matrix

$$(\mu(q)^i{}_j) = \begin{pmatrix} a & \alpha b & \alpha c & -d \\ b & a & -\alpha d & \alpha c \\ c & \alpha d & a & -\alpha b \\ d & c & -b & a \end{pmatrix}$$

Lowering index of endomorphism $\mu(q)$ we get the tensor of the type (2,0) on the real space $\mathbf{H}_\alpha$ that will be skew-symmetric iff $a = 0$. Thus, lowering index of the purely imaginary α-quaternion $q \in \{\mathsf{Q}\}$ regarded in canonical representation of the second kind and using the fibre metric we get 2-form on the space of fibre bundle Q that we call *a pseudo-fundamental form of the α-quaternion.*

Similarly to the above, an AQ_α-manifold M is assumed calibrated and thus, $\mathsf{X}(\mathsf{Q}) = \mathcal{V}\oplus\mathcal{H}$, where $\mathcal{V}$ and $\mathcal{H}$ are, respectively, vertical and horisontal distributions of metric connection in fibre bundle Q induced by calibration. Recall that tensor $t \in \mathcal{T}^s_r(\mathsf{Q})$ is called *vertical* if it vanishes if at least one of the arguments of the tensor regarded as a polylinear function is horisontal. In other word, a tensor t is vertical if $\forall\, X_1, \ldots, X_r \in \mathsf{X}(\mathsf{Q})$, $\forall\,\omega^1, \ldots, \omega^s \in \mathsf{X}^*(\mathsf{Q}) \Longrightarrow t(X_1, \ldots, X_r, \omega^1, \ldots, \omega^s) = t(X_1^V, \ldots, X_r^V, \omega^1_V, \ldots, \omega^s_V)$, where X^V, ω_V are vertical constituents of vector X and covector ω, respectively. Vertical tensors define a subalgebra of tensor algebra $\mathcal{T}(\mathsf{Q})$ that we call *vertical tensor algebra* and denote it by $\mathcal{T}_V(\mathsf{Q})$. Similar remarks refer to Grassnanian algebra $\Lambda(\mathsf{Q}) \supset \Lambda_V(\mathsf{Q})$. In particular, pointwise localization of module $(\Lambda^r)_V(\mathsf{Q})$ has dimension $C_4^r = \frac{4!}{r!(4-r)!}$; $\quad 0 \le r \le 4$, and zero dimension, when $r > 4$. Thus, we have naturally defined the Grassmanian algebra $\Lambda_V(\mathsf{Q}) = \oplus_{r=0}^4 (\Lambda^r)_V(\mathsf{Q})$ that we call *a vertical Grassmanian algebra.* Note that module $(\Lambda^4)_V$ is one-dimensional; as its basis we naturally take the form η defined by the equality

$$\eta_p = \sqrt{\det(g)}\,\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3,$$

where (g) is Gramme matrix of metric g_p, $\{\,p, \omega^0, \omega^1, \omega^2, \omega^3\,\}$ is a coframe dual to frame $\{\,p, e_0, e_1, e_2, e_3\,\} \in B\mathsf{Q}$ of the space $\mathcal{V}_p = \mathsf{Q}_p$; $\quad p \in \mathsf{Q}$. We check the independence of η_p on the choice of basis in a standard way. We call the form η *a vertical volume form.* Evidently, it generates the orientation of the bundle $\mathcal{V} = \{\mathsf{Q}\}$.

Definition 18. We call *(vertical) Hodge operator* the endomorphism $* : \Lambda_V(\mathsf{Q}) \to \Lambda_V(\mathsf{Q})$ being defined by the equality

$$\omega\wedge(*\vartheta) = (\omega, \vartheta)\eta; \quad \omega, \vartheta \in (\Lambda^r)_V(\mathsf{Q}), \quad r = 0, 1, 2, 3, 4;$$

where $(\cdot,\cdot)$ is scalar product in $\Lambda_V(\mathsf{Q})$ induced by metric in Q.

In particular, $* : (\Lambda^2)_V(\mathsf{Q}) \to (\Lambda^2)_V(\mathsf{Q})$, and $(\omega,\vartheta) = \frac{1}{2}\omega_{\beta\gamma}\,\vartheta^{\beta\gamma}$, then $(*\omega)_{\beta\gamma} = \omega_{\delta\varepsilon}$, where $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \beta & \gamma & \delta & \varepsilon \end{pmatrix}$ is even permutation. Hence, $(*)^2 = \mathrm{id}$ on $(\Lambda^2)_V(\mathsf{Q})$ and then, $*|_{(\Lambda^2)_V(\mathsf{Q})}$ has eigenvalues ± 1.

Definition 19. A vertical 2-form on Q is called *self-dual* (resp., *anti-self-dual*) if it is an eigenvector of Hodge operator with eigenvalue 1 (resp., -1). If it is also parallel along the fibres of structural bundle (i.e. projectable) it is called *a form on M.* The module of self-dual (resp., anti-self-dual) forms on M will be denote by $\Lambda^+(M)$ (resp., $\Lambda^-(M)$).

Theorem 23. *A projectable vertical 2-form is self-dual on an AQ_α-manifold M iff it is a fundamental form of a purely imaginary α-quaternion. Here, the orientation of fibre bundle Q generated by exterior square of the form corresponds to the canonical orientation of the bundle iff the α-quaternion has a real norm.*

Proof. Let ω be a projectable vertical 2-form. By definition, $\omega \in \Lambda^+(M) \iff *\omega = \omega$. Denote by $\{\,\eta_{\beta\gamma\delta\varepsilon}\,\}$ components of tensor η in a positively oriented frame. Then, if $\{\,\omega_{\beta\gamma}\,\}$ are components of the 2-form ω, then the condition of its self-duality (see (7)) can be written in the form

$$\omega_{\beta\gamma} = \frac{1}{2}\eta_{\beta\gamma\delta\varepsilon}\, g^{\delta\zeta}\, g^{\varepsilon\vartheta}\, \omega_{\zeta\vartheta}.$$

On the space of bundle $B\mathsf{Q}$, where $(g_{\beta\gamma}) = diag(1,-\alpha,-\alpha,1)$ these correlations will assume the form $\omega_{\hat\beta\hat\gamma} = \varepsilon(\beta,\gamma)\,\omega_{\beta\gamma}$, where $\varepsilon(\beta,\gamma) = g_{\beta\beta}\, g_{\gamma\gamma}; \quad \beta,\gamma = 0,1,2,3$; $(\hat\beta,\hat\gamma)$ is a complementing the pair (β,γ) up to the even permutation of indices (0,1,2,3). It means that

$$\omega \in \Lambda^+(M) \iff (\omega_{\beta\gamma}) = \begin{pmatrix} 0 & \alpha x & \alpha y & -z \\ -\alpha x & 0 & -z & y \\ -ay & z & 0 & -x \\ z & -y & x & 0 \end{pmatrix} \tag{38}$$

Raising the index of the form we get the endomorphism of the fibre bundle Q that, by the above, can be identified with the purely imaginary α-quaternion $q = xJ_1 + yJ_2 + zJ_3$. Evidently, the inverse is also true: by (38) the fundamental form of purely imaginary α-quaternion is self-dual. Note that, by definition of self-dual form, $\omega\wedge\omega = \omega\wedge(*\omega) = (\omega,\omega)\eta$, and hence, orientation of the fibre bundle Q generated by the form $\omega\wedge\omega$ coincides with the canonical orientation of this fibre bundle iff $\|\omega\|^2 > 0$, i.e. $|q|^2 > 0$. $\square$

Similarly, we prove

Theorem 24. *A projectable vertical 2-form is anti-self-dual on a AQ_α-manifold iff it is a pseudofundamental form of a purely imaginary α-quaternion. Here, orientation of the fibre bundle generated by exterior square of the form is inverse to the natural orientation of the fibre bundle iff the α-quaternion has a real norm.* $\square$

4.2 Vertical tensors on AQ_α-manifold

Let M be an $4n$-dimensional almost α-quaternionic manifold, Q be its structural bundle. In a standard way [4] it is proved that giving of an AQ_α-structure on M is equivalent to giving of $\mathcal{G}$-structure on the manifold with structural group $\mathcal{G} = GL(n, \mathbf{H}_\alpha) \cdot Sp_\alpha(1)$, where $Sp_\alpha(m)$ is a symplectic group of order m over the ring $\mathbf{H}_\alpha$. The elements of the space of this $\mathcal{G}$-structure called *adapted frames*, or *A-frames*, are constructing in the following way. Let $p \in M$, $r = (J_0 = \mathrm{id}, J_1, J_2, J_3) \in B\mathsf{Q}$ be a positively oriented orthonormalized basis of fibre Q_p, $(e_1, \dots, e_n)$ be the basis of the space $T_p(M)$ regarded as $\mathbf{H}_\alpha$-module: if $q = a + bj_1 + cj_2 + dj_3 \in \mathbf{H}_\alpha, \quad X \in T_p(M)$, then $qX = aX + bJ_1X + cJ_2X + dJ_3X$. Assume the greek indices to be in range from 0 to 3, the latin ones - in range from 1 to n. Denote $e_{\beta a} = J_\beta(e_a)$. Then $(p, e_{\beta a}; \quad \beta = 0, \dots, 3; \quad a = 1, \dots, n)$ is a frame of the space $T_p(M)$ called *adapted.* Evidently, in this frame

$$(J_1) = \begin{pmatrix} 0 & \alpha I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I \\ 0 & 0 & I & 0 \end{pmatrix}; \qquad (J_2) = \begin{pmatrix} 0 & 0 & \alpha I & 0 \\ 0 & 0 & 0 & -\alpha I \\ I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix};$$
$$(J_3) = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & \alpha I & 0 \\ 0 & -\alpha I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

and then $(J_k) = (j_k) \otimes I; \quad k = 1, 2, 3; \quad I$ is a unit matrix of order n. Evidently, giving tensor t of the type (1,1) on M is defined by giving of a function set $\{ t^{\beta i}_{\gamma j} \}$ on the space of $\mathcal{G}$-structure that are components of the tensor in the corresponding A-frame. It follow from (39) that such tensor is an α-quaternion $q = a\,\mathrm{id} + bJ_1 + cJ_2 + dJ_3$ iff $t^{\beta i}_{\gamma j} = q^\beta_\gamma\, \delta^i_j$, where

$$(q^\beta_\gamma) = \begin{pmatrix} a & \alpha b & \alpha c & -d \\ b & a & \alpha d & -\alpha c \\ c & -\alpha d & a & \alpha b \\ d & -c & b & a \end{pmatrix}$$

is the matrix of α-quaternion q in basis $\{ \mathrm{id}, J_1, J_2, J_3 \}$ regarded as endomorphism of the structural bundle.

Consider the natural representation of endomorphisms algebra of module $\mathsf{X}(M)$ of AQ_α-manifold M into endomorphisms algebra of module $\mathcal{T}_1^1(M)$, the representation being generated by the left shifts. Namely, if $f \in \mathcal{T}_1^1(M)$ we juxtapose it to endomorphism $\hat{f}$ acting by formula $\hat{f}(g) = f \circ g; \quad g \in \mathcal{T}_1^1(M)$. Evidently, the mapping $f \to \hat{f}$ is representation in view of endomorphisms algebra being associative. The question naturally arises: when does element $f \in \mathcal{T}_1^1(M)$ in this representation preserves the structural bundle and, thus, induces the structural bundle endomorphism ? The question is answered by

Theorem 25. *Let M be an AQ_α-manifold. Endomorphism $t \in \mathcal{T}_1^1(M)$ induces structural bundle endomorphism iff its components on the space of $\mathcal{G}$-structure have the form*

$$t^{\gamma c}_{\beta b} = t^\gamma_\beta \, \delta^c_b. \tag{40}$$

Proof. Let equalities (40) hold and let $q \in \{\, \mathsf{Q}\,\}$. Assume $\hat{t}(q) = t \circ q$. Fix frame $(p, J_\beta; \quad \beta = 0, \ldots, 3)$ in Q_p and its corresponding A-frame $(p, e_{\beta b}) \in \mathcal{G}$. We have $\hat{t}(J_\beta)(e_b) = t \circ J_\beta(e_b) = t(J_\beta(e_b)) = t(e_{\beta b}) = t^{\gamma c}_{\beta b}\, e_{\gamma c} = t^\gamma_\beta \, \delta^c_b \, e_{\gamma c} = t^\gamma_\beta \, e_{\gamma b} = t^\gamma_\beta \, J_\gamma(e_b); \quad b = 1, \ldots, n$, hence $\hat{t}(J_\beta) = t^\gamma_\beta \, J_\gamma$, i.e. endomorphism $\hat{t} : f \to t \circ f$ of module $\mathcal{T}_1^1(M)$ preserves the structural bundle, and thus, induces its endomorphism. Inversely, let $\hat{t} : f \to t \circ f$ preserves the structural bundle. Then $\hat{t}(J_\beta) = t^\gamma_\beta \, J_\gamma$ and $t(e_{\beta b}) = t(J_\beta(e_b)) = t \circ J_\beta(e_b) = \hat{t}(J_\beta)(e_b) = t^\gamma_\beta \, J_\gamma(e_b) = t^\gamma_\beta \, \delta^c_b \, J_\gamma(e_c) = t^\gamma_\beta \, \delta^c_b \, e_{\gamma c}$. Thus, $t^{\gamma c}_{\beta b} = t^\gamma_\beta \, \delta^c_b$. $\square$

Evidently, giving structural bundle endomorphism is equivalent to giving vertical tensor $t_V \in (\mathcal{T}_1^1)_V(\mathsf{Q})$ and, thus, Theorem 25 can be formulated as follow:

Theorem 26. *Let M be an AQ_α-manifold. Endomorphism $t \in \mathcal{T}_1^1(M)$ in natural representation generates vertical tensor $t_V \in (\mathcal{T}_1^1)_V(\mathsf{Q})$ iff its components on the space of $\mathcal{G}$-structure have the form $t^{\gamma c}_{\beta b} = t^\gamma_\beta \, \delta^c_b$.* $\square$

Definition 20. Endomorphism t of the module $\mathsf{X}(M)$ of a AQ_α-manifold M preserving the structural bundle in the natural representation and, thus, generating vertical tensor on Q is called *vertical endomorphism.*

Example. As we have seen, any α-quaternion $q \in \{\, \mathsf{Q}\,\}$ satisfies the condition of Theorem 26 and, thus, it is a vertical endomorphism. The validity of (40) in this case can also be shown in another way. Let $q \in \mathsf{Q}$. Then $q(e_{\beta b}) = q(J_\beta(e_b)) = q^\gamma \, J_\gamma(J_\beta(e_b)) = q^\gamma (J_\gamma \circ J_\beta) e_b = q^\gamma \, C^\varepsilon_{\gamma\beta} \, J_\varepsilon(e_b) = q^\gamma \, C^\varepsilon_{\gamma\beta} \, \delta^c_b \, J_\varepsilon(e_c) = q^\varepsilon_\beta \, \delta^c_b \, e_{\varepsilon c}$ i.e. $q^{\varepsilon c}_{\beta b} = q^\varepsilon_\beta \, \delta^c_b$, where $q^\varepsilon_\beta = C^\varepsilon_{\gamma\beta} \, q^\gamma$, $\{\, C^\varepsilon_{\gamma\beta}\,\}$ are structural tensor components of α-quaternion algebra.

Similar terminology will be preserved for other tensor types on M, received from vertical endomorphisms by means of classical operations of tensor algebra.

In particular, if M is a pseudo-Riemannian manifold, a 2-form on M is called *vertical* if it is corresponds to a vertical endomorphism in raising the index. Evidently, the set of all vertical 2-forms on M forms a submodule $(\Lambda^2)_V(M) \subset \Lambda^2(M)$ that we call *a module of vertical 2-forms on M.*

Definition 21. An AQ_α-structure on a pseudo-Riemannian manifold $(M, G = \langle\cdot,\cdot\rangle)$ is called *generalized quaternionic-Hermitian*, or *$\mathcal{H}AQ_\alpha$-structure* if

$$\forall J \in \{\,\mathsf{T}\,\} \Longrightarrow \langle JX, Y\rangle + \langle X, JY\rangle = 0; \qquad X, Y \in \mathsf{X}(M).$$

In this case for any purely imaginary α-quaternion q on M tensor $\Omega(X,Y) = \langle X, qY\rangle$ is a vertical form on M called *Kaehler form* of α-quaternion q. Kaehler 2-forms on M form a submodule $\mathcal{K}(M) \subset (\Lambda^2)_V(M)$ that we call *Kaehler module* of $\mathcal{H}AQ_\alpha$-structure. By Theorem 23, the image of Kaehler module in natural representation (in combination with lowering the index by means of fibre metric) is a module of self-dual forms on M.

Giving an $\mathcal{H}AQ_\alpha$-structure on manifold M^{4n} is equivalent to giving $\mathcal{G}$-structure on M with structural group $\mathcal{G} = Sp_\alpha(n)\cdot Sp_\alpha(1)$ whose total space elements are A-frames, and their vectors $\{\,e_1,\dots,e_n\,\}$ form an orthonormalized system in quaternionic-Hermitian metric $\langle\langle X,Y\rangle\rangle = \langle X,Y\rangle + i\langle X, I^3Y\rangle + j\langle X, J^3Y\rangle + k\langle X, K^3Y\rangle$, where $\{\,\mathrm{id}, I, J, K\,\}$ is positively oriented orthonormalized basis of fibre Q_p an arbitrary point $p \in M$.

Lemma 3. *On the space of $\mathcal{G}$-structure components of metric tensor of an $\mathcal{H}AQ_\alpha$-manifold have the form: $G_{\beta b\gamma c} = g_{\beta\gamma}\,G_{bc}$, where $\{\,g_{\beta\gamma}\,\}$ are components of fibre metric of the structural bundle, $G_{bc} = \langle e_b, e_c\rangle$.*

Proof. We have:

$$\begin{aligned} G_{\beta b\gamma c} &= \langle J_\beta(e_b), J_\gamma(e_c)\rangle = -\langle J_\gamma \circ J_\beta(e_b), e_c\rangle = \\ &= -C^\delta_{\gamma\beta}\langle J_\delta(e_b), e_c\rangle = -C^0_{\gamma\beta}\langle e_b, e_c\rangle, \end{aligned}$$

where $\{\,C^\delta_{\gamma\beta}\,\}$ are structural constants of α-quaternion algebra. Note that since for any pair $\{\,q_1, q_2\,\}$ of purely imaginary α-quaternions $q_1q_2 + q_2\,q_1 = -2\langle q_1, q_2\rangle$, $q_1q_2 - q_2\,q_1 = [q_1, q_2]$, then $q_1\,q_2 = -\langle q_1, q_2\rangle + \frac{1}{2}[q_1, q_2]$. Thus, $\mathrm{Re}\,(q_1q_2) = -\langle q_1, q_2\rangle$, $\mathrm{Im}\,(q_1q_2) = \frac{1}{2}[q_1, q_2]$, where $\mathrm{Re}\,q$ and $\mathrm{Im}\,q$ are real and purely imaginary parts of α-quaternion q, respectively. Hence, $C^0_{\gamma\beta} = \mathrm{Re}\,(J_\gamma J_\beta) = -\langle J_\gamma, J_\beta\rangle = -g_{\gamma\beta} = -g_{\beta\gamma}$. Thus, $G_{\beta b\gamma c} = g_{\beta\gamma}\langle e_b, e_c\rangle = g_{\beta\gamma}\,G_{bc}$. □

Let Ω be a vertical 2-form on an $\mathcal{H}AQ_\alpha$-manifold M, $\{\,\Omega_{\beta b\gamma c}\,\}$ be its components on the space of $\mathcal{G}$-structure. By Theorem 26, $G^{\varepsilon h\beta b}\,\Omega_{\varepsilon h\gamma c} = t^\beta_\gamma\,\delta^b_c$, and thus, $G_{\beta b\delta d}\,G^{\varepsilon h\beta b}\,\Omega_{\varepsilon h\gamma c} = t^\beta_\gamma\,G_{\beta c\delta d}$. But $G_{\beta b\delta d}\,G^{\varepsilon h\beta b} = \delta^\varepsilon_\delta\,\delta^h_d$, hence, $\Omega_{\delta d\gamma c} = G_{\delta d\beta c}\,\delta^\beta_\gamma$. But by Lemma 3,

$$\Omega_{\delta d\gamma c} = \omega_{\delta\gamma}\,G_{dc}, \tag{41}$$

where $\omega_{\delta\gamma} = g_{\delta\beta}\, t^{\beta}_{\gamma}$ are components of tensor ω on Q, being the image of form Ω in natural representation. From (41) it follows that $\omega_{\delta\gamma} = \frac{1}{n} G^{dc}\, \Omega_{\delta d \gamma c}$, in particular, ω is skew-symmetric tensor being, evidently, a projectable vertical 2-form on the space Q. Evidently, the inverse is also true: let ω be projectable vertical 2-form on Q, then 2-form Ω on M defined by (41) is a vertical 2-form on M. Thus, we have proved

Theorem 27. *The module of vertical 2-forms on an $\mathcal{H}AQ_\alpha$-manifold M in natural representation coincides with module $(\Lambda_{\pi V})^2(\mathsf{Q})$ of projectable vertical 2-forms on the space Q.* $\square$

Now we shall identify the moduls sometime.

As a corollary we get the following result:

Theorem 28. *The module of vertical 2-forms on an $\mathcal{H}AQ_\alpha$-manifold M is decomposed into orthogonal direct sum of Kaehler module and the submodule coinciding in natural representation with the submodule of anti-self-dual forms on M and serving as a Kaehler module of a uniquely defined $\mathcal{H}AQ_\alpha$-structure on M.*

Proof. Note that module $(\Lambda_{\pi V})^2(\mathsf{Q})$ of projectable vertical 2-forms on Q is decomposed into orthogonal direct sum of submodules of self-dual and anti-self-dual forms on M. Indeed, by definition, $(\Lambda_{\pi V})^2(\mathsf{Q}) = \Lambda^+(M) \oplus \Lambda^-(M)$.If $\omega \in \Lambda^+(M), \vartheta \in \Lambda^-(M)$, then $\langle \omega, \vartheta \rangle \eta = \omega \wedge (*\vartheta) = -\omega \wedge \vartheta = -\vartheta \wedge \omega = -\vartheta \wedge (*\omega) = -\langle \vartheta, \omega \rangle \eta = -\langle \omega, \vartheta \rangle \eta$, hence, $\langle \omega, \vartheta \rangle = 0$, and thus, $\Lambda^+(M) \perp \Lambda^-(M)$. Now let $\Omega, \Theta \in \Lambda_V(M)$, ω, ϑ are their images in natural representation. From (41) it follows that the forms Ω and Θ are orthogonal in metric G iff the forms ω and ϑ are orthogonal in fibre metric. In particular, submodule $\mathcal{K}(M)^\perp$ in natural representation coincides with submodule of anti-self-dual forms on M. Further, as in proof of Theorem 4, we see that in natural identification of elements of module $\mathcal{K}(M)^\perp$ with endomorphisms of module $\mathsf{X}(M)$ fibre bundle $\mathcal{S}c(M) \oplus \mathcal{K}(M)^\perp$ defines an $\mathcal{H}AQ_\alpha$-structure on M. $\square$

Definition 22. $\mathcal{H}AQ_\alpha$-structures $\mathsf{Q} = \mathcal{S}c(M) \oplus \mathcal{K}(M)$ and $\mathsf{Q}^* = \mathcal{S}c(M) \oplus \mathcal{K}(M)^\perp$ on manifold M will be called *conjugate.*

4.3 Generalized quaternionic-Kaehler structures

Definition 23. An $\mathcal{H}AQ_\alpha$-structure is called *a generalized quaternionic-Kaehler*, or $\mathcal{K}AQ_\alpha$-structure if its structural bundle is invariant with respect to parallel translations in Riemannian connection.

It means that Riemannian connection ∇ is an AQ_α-connection, and we fix it as calibration.

Evidently, any quaternionic-Kaehler structure (see example 3 in section 2.2) is a $\mathcal{K}AQ_\alpha$-structure.

Theorem 29. *A kq-manifold M of dimension $4n > 4$ is an Einsteinian manifold.*

Proof. Let $\mathsf{U} = \{ U_a \}_{a\in A}$ be the local triviality covering of $\mathcal{G}$-structure for manifold M, $U \in \mathsf{U}$. Then narrowing Q on U is a πAQ_α-structure, and thus, there is a pair twistors $\{ I, J \}$ on U, such that $\{ \mathsf{Q}|U \}$ is generated by their algebraic shell, i.e. $\{ \mathsf{Q}|U \} = \mathcal{L}(\mathrm{id}, J_1, J_2, J_3)$, where $J_1 = I, J_2 = J, J_3 = K = I \circ J$. Besides, there exists the system $\{ e_1, \ldots, e_n \}$ of vector fields on U such that the system $\{ e_{\beta b} \}$, $\beta = 0, \ldots, 3$; $b = 1, \ldots, n$, where $e_{\beta b} = J_\beta(e_b)$, defines the space section of $\mathcal{G}$-structure over U. In this case if $X, Y \in \mathsf{X}(U)$ then $\nabla_Y (J_\beta) = \omega(Y)^\gamma_\beta J_\gamma$, where $\omega = \{ \omega^\gamma_\beta \}$ is a form on U with values in Lie algebra $\mathsf{so}(2-\alpha, 1+\alpha; \mathbf{R})$ of the structural group of bundle $B\mathsf{Q}$, at every point of U. Further, let $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is a Riemann-Christoffel tensor. Then

$$\begin{aligned}
R(X,Y)(J_\beta) &= \nabla_X \nabla_Y (J_\beta) - \nabla_Y \nabla_X (J_\beta) - \nabla_{[X,Y]}(J_\beta)\\
&= \nabla_X (\omega(Y)^\gamma_\beta J_\gamma) - \nabla_Y (\omega(X)^\gamma_\beta J_\gamma) - \omega([X,Y])^\gamma_\beta J_\gamma\\
&= X\omega(Y)^\gamma_\beta J_\gamma + \omega(Y)^\gamma_\beta \nabla_X J_\gamma - Y\omega(X)^\gamma_\beta J_\gamma -\\
&\quad -\omega(X)^\gamma_\beta \nabla_Y J_\gamma - \omega([X,Y])^\gamma_\beta J_\gamma\\
&= (X\omega(Y)^\gamma_\beta - Y\omega(X)^\gamma_\beta - \omega([X,Y])^\gamma_\beta) J_\gamma +\\
&\quad +\omega(Y)^\gamma_\beta \omega(X)^\delta_\gamma J_\delta - \omega(X)^\gamma_\beta \omega(Y)^\delta_\gamma J_\delta\\
&= 2d\omega(X,Y)^\gamma_\beta J_\gamma + (\omega(X)^\delta_\gamma \omega(Y)^\gamma_\beta - \omega(Y)^\delta_\gamma \omega(X)^\gamma_\beta) J_\delta\\
&= 2(d\omega(X,Y)^\gamma_\beta + \frac{1}{2}[\omega,\omega](X,Y)^\gamma_\beta) J_\gamma.
\end{aligned}$$

Thus,

$$R(X,Y)J_\beta = 2D\omega(X,Y)^\gamma_\beta J_\gamma; \qquad \beta, \gamma = 1,2,3, \tag{42}$$

where $D\omega = d\omega + \frac{1}{2}[\omega, \omega]$ is a 2-form on U with values in Lie algebra $\mathsf{so}(2-\alpha, 1+\alpha; \mathbf{R})$. Here $R(X,Y)$ is regarded as differentiation of tensor algebra on manifold M generated by endomorphism $R(X,Y)$ of module $\mathsf{X}(M)$, and thus,

$$R(X,Y)J_\beta = [R(X,Y), J_\beta]. \tag{43}$$

Let the form $D\omega$ be given by the function matrix

$$(D\omega^\gamma_\beta) = \begin{pmatrix} 0 & -c & b \\ & 0 & -a \\ \alpha b & -\alpha a & 0 \end{pmatrix}.$$

Then by (43) identities (42) can be written in the form

$$\begin{aligned}
&1)\ [R(X,Y), I] = && c(X,Y)J + \alpha b(X,Y)K;\\
&2)\ [R(X,Y), J] = -c(X,Y)I && - \alpha a(X,Y)K;\\
&3)\ [R(X,Y), K] = b(X,Y)I - a(X,Y)J. &&
\end{aligned} \tag{44}$$

Apply (44_3) to vector $Z \in \mathsf{X}(U)$ and find scalar product of the result and vector JZ:

$$\begin{aligned}\langle R(X,Y)KZ, JZ\rangle - \langle K \circ R(X,Y)Z, JZ\rangle = \\ = b(X,Y)\langle IZ, JZ\rangle - a(X,Y)\langle JZ, JZ\rangle.\end{aligned} \tag{45}$$

Since $\langle IZ, JZ\rangle = -\langle Z, KZ\rangle = 0$; $\langle JZ, JZ\rangle = -\alpha\langle Z, Z\rangle$; $\langle R(X,Y)KZ, JZ\rangle = -\langle R(X,Y)JZ, KZ\rangle$; $-\langle K \circ R(X,Y)Z, JZ\rangle = \langle J \circ I \circ R(X,Y)Z, JZ\rangle = -\alpha\langle I \circ R(X,Y)Z, Z\rangle = \alpha\langle R(X,Y)Z, IZ\rangle$, equality (45) can be rewritten in the form

$$a(X,Y)\|Z\|^2 = \langle R(X,Y)Z, IZ\rangle - \alpha\langle R(X,Y)JZ, KZ\rangle. \tag{46}$$

Numerate elements $\{ e_{\beta b} \}$ of the local section of $\mathcal{G}$-structure space by one index $\{ e_i \}$, $\quad i = 1, \dots, 4n$. Assume $Z = \|e_i\| e_i$ in (33) and summarize by i from 1 to $4n$. Remark that $-\alpha \sum_{i=1}^{4n} \langle R(X,Y)Je_i, Ke_i\rangle \|e_i\|^2 = \sum_{i=1}^{4n} \langle R(X,Y)Je_i, I \circ Je_i\rangle \|Je_i\|^2 = \sum_{i=1}^{4n} \langle R(X,Y)e_i, Ie_i\rangle \|e_i\|^2$, and $\|e_i\|^4 = 1$. Then we get:

$$4na(X,Y) = 2\sum_{i=1}^{4n} \langle R(X,Y)e_i, Ie_i\rangle \|e_i\|^2. \tag{47}$$

Apply Ricci identity to the right part of the equality:

$$2na(X,Y) = -\sum_{i=1}^{4n} \{\langle R(X,e_i)Ie_i, Y\rangle + \langle R(X,Ie_i)Y, e_i\rangle\} \|e_i\|^2,$$

or

$$2na(X,Y) = \sum_{i=1}^{4n} \{\langle R(X,e_i)Y, Ie_i\rangle - \langle R(X,Ie_i)Y, e_i\rangle\} \|e_i\|^2.$$

Note that

$$\begin{aligned}\sum_{i=1}^{4n} \langle R(X,Ie_i)Y, e_i\rangle \|e_i\|^2 &= \sum_{i=1}^{4n} \langle R(X,I^2 e_i)Y, Ie_i\rangle \|Ie_i\|^2 \\ &= -\sum_{i=1}^{4n} \langle R(X,e_i)Y, Ie_i\rangle \|e_i\|^2,\end{aligned}$$

and then,

$$na(X,Y) = \sum_{i=1}^{4n} \langle R(X,e_i)Y, Ie_i\rangle \|e_i\|^2 = -\sum_{i=1}^{4n} \langle I \circ R(X,e_i)Y, e_i\rangle \|e_i\|^2.$$

But in view of (44_1),

$$\begin{aligned}-\sum_{i=1}^{4n} \langle R(X,e_i)Y, e_i\rangle \|e_i\|^2 = -\sum_{i=1}^{4n} \{ \langle R(X,e_i)IY, e_i\rangle - c(X,e_i)\langle JY, e_i\rangle - \\ -\alpha b(X,e_i)\langle KY, e_i\rangle \} \|e_i\|^2 = -Ric(X,IY) + c(X,JY) + \alpha b(X,KY),\end{aligned}$$

hence,

$$na(X,Y) = -Ric(X,IY) + c(X,JY) + \alpha b(X,KY). \tag{48}$$

Where Ric is Ricci tensor. Similarly,

$$nb(X,Y) = -Ric(X,JY) - \alpha a(X,KY) - c(X,IY), \tag{49}$$

$$nc(X,Y) = \alpha\, Ric(X,KY) - \alpha b(X,IY) + \alpha a(X,JY). \tag{50}$$

Substituting IY, JY and KY for Y in (48),(49) and (50), respectively, and in view of $I^2 = J^2 = \alpha\,\mathrm{id}$, $K^2 = -\,\mathrm{id}$, we get:

$$\begin{aligned} na(X,JY) + b(X,JY) + c(X,KY) &= -\alpha\, Ric(X,Y); \\ a(X,IY) + nb(X,JY) + c(X,KY) &= -\alpha\, Ric(X,Y); \\ a(X,IY) + b(X,JY) + nc(X,KY) &= -\alpha\, Ric(X,Y). \end{aligned} \tag{51}$$

Substracting elementwise equations of the system, we get:

$$a(X,IY) = b(X,JY) = c(X,KY),$$

and in view of (51),

$$a(X,IY) = b(X,JY) = c(X,KY) = -\frac{\alpha}{n+2}\, Ric(X,Y). \tag{52}$$

Now we can rewrite (46) in the form

$$\langle R(X,IY)Z,IZ\rangle - \alpha\langle R(X,IY)JZ,KZ\rangle = -\frac{\alpha}{n+2}\, Ric(X,Y)\|Z\|^2.$$

In particular, if $X = Y$, the equality has the form

$$\langle R(X,IX)Z,IZ\rangle - \alpha\langle R(X,IX)JZ,KZ\rangle = -\frac{\alpha}{n+2}\, Ric(X,X)\|Z\|^2. \tag{53}$$

Substitute JX for X in (53):

$$\langle R(JX,KX)Z,IZ\rangle - \alpha\langle R(JX,KX)JZ,KZ\rangle = \frac{\alpha}{n+2}\, Ric(JX,JX)\|Z^2. \tag{54}$$

Note that

$$Ric(JX,JX) = -\alpha(n+2)b(JX,J^2X) = -(n+2)b(JX,X) = -\alpha\, Ric(X,X).$$

In view of above multiply both parts of (54) by $(-\alpha)$ and add elementwise to (53):

$$\begin{aligned} &\langle R(X,IX)Z,IZ\rangle - \alpha\langle R(X,IX)JZ,KZ\rangle - \alpha\langle R(JX,KX)Z,IZ\rangle + \\ &\qquad + \langle R(JX,KX)JZ,KZ\rangle = -\frac{2\alpha}{n+2}\, Ric(X,X)\|Z\|^2. \end{aligned}$$

Note that the left part of the equality is symmetric with respect to X and Z. Thus, $Ric(X,X)\|Z\|^2 = Ric(Z,Z)\|X\|^2$. Polarize this equality by argument X, we get: $Ric(X,Y)\|Z\|^2 = Ric(Z,Z)\langle X,Y\rangle$. Assume here $Z = \|e_i\|e_i$ and summarize by i from 1 to $4n$: $4n\,Ric(X,Y) = s\langle X,Y\rangle$, where s is scalar curvature of the manifold. Thus,

$$Ric(X,Y) = \frac{s}{4n}\langle X,Y\rangle,$$

i.e. M is Einsteinian manifold. $\square$

The above result generalizes the well-known Berger theorem of quaternionic-Kaehler manifolds of dimension greater than 4 being Einsteinian [1].

4.4 $\mathcal{H}AQ_\alpha$-structures of vertical type

Let M be an $\mathcal{H}AQ_\alpha$-manifold, R be a Riemann-Christoffel tensor of metric G. The classical features of symmetry of this tensor mean that it can be regarded as self-conjugate endomorphism of a 2-form module on M. It will be called a *Riemann-Cristoffel endomorphism.*

Definition 24. An $\mathcal{H}AQ_\alpha$-structure on manifold M is called *the structure of vertical type* if a Riemann-Christoffel endomorphism generated by the manifold metric preserves the module of vertical 2-forms on M.

For example, by Theorem 26 a canonical $\mathcal{H}AQ_\alpha$-structure of a 4-dimensional oriented pseudo-Riemannian manifold has a vertical type. Moreover, there holds the following

Theorem 30. *Any $\mathcal{K}AQ_\alpha$-manifold is a $\mathcal{H}AQ_\alpha$-manifold of vertical type.*

Proof. By definition, the structural bundle of $\mathcal{K}AQ_\alpha$-manifold M, $\dim M = 4n$, is invariant with respect to parallel translations in Riemannian connection ∇ of manifold. The module of vertical endomorphisms as well as the one of vertical 2-forms on M have, naturally, the same property. (Indeed, if $t \in \mathcal{T}^1_1(M)$ is a vertical endomorphism, $X \in \mathsf{X}(M)$, then $(\nabla_X\,\hat{t})q = \nabla_X\,(t\circ q) - t\circ\nabla_X\,(q) \in \{\,\mathsf{Q}\,\};\quad q\in\{\,\mathsf{Q}\,\}$). Moreover, in view of the parallelity of the metric tensor in Riemannian connection the $\mathcal{H}AQ_\alpha$-structure conjugate to a $\mathcal{K}AQ_\alpha$-structure is a $\mathcal{K}AQ_\alpha$-structure itself. Now let $n>1$, I be a twistor at an arbitrary point $p\in M$, Ω be its Kaehler form. Then in view of (47) and (52) we have

$$\begin{aligned}
R(\Omega)(X,Y) &= \frac{1}{2}R(X,Y)_{ij}\Omega^{ij}\\
&= \frac{1}{2}\sum_{i,j=1}^{4n}\langle R(X,Y)e_i,e_\rangle\Omega(e_i,e_j) = \frac{1}{2}\sum_{i,j=1}^{4n}\langle R(X,Y)e_i,e_j\rangle\langle e_i,Ie_j\rangle\\
&= \frac{1}{2}\sum_{i,j=1}^{4n}\langle R(X,Y)e_i,Ie_j\rangle\langle e_i,I^2e_j\rangle = \frac{\alpha}{2}\sum_{i=1}^{4n}\langle R(X,Y)e_i,Ie_i\rangle\|e_i\|^2
\end{aligned}$$

$$= \alpha n a(X,Y) = na(X, I^2 Y) = -\frac{\alpha n}{n+2} Ric(X, IY),$$

where Ric is Ricci tensor of manifold M, $\quad X, Y \in \mathsf{X}(M)$. Here $\{ e_1, \ldots, e_{4n} \}$ is orthonormalized bazis adapted to twistor J. But by Theorem 29, a $\mathcal{K}AQ_\alpha$-manifold of dimension $4n > 4$ is Einsteinian and thus,

$$R(\Omega)(X,Y) = -\frac{\alpha n \varepsilon}{n+2} \langle X, IY \rangle = -\frac{\alpha n \varepsilon}{n+2} \Omega(X,Y),$$

i.e.

$$R(\Omega) = -\frac{\alpha n \varepsilon}{n+2} \Omega, \tag{55}$$

where ε is Einstein constant. In particular, endomorphism R transfers the Kaehler module of a $\mathcal{K}AQ_\alpha$-structure into itself. But in view of the same considerations this endomorphism transfers into itself the Kaehler module of the conjugate structure, too. And by Theoreme 28 the endomorphism preserves the module of vertical 2-forms on M. $\square$

Corollary. *Any quaternionic-Kaehler manifold is an $\mathcal{H}AQ_\alpha$-manifold of vertical type.* $\square$

Let M be an $\mathcal{H}AQ_\alpha$-manifold of vertical type. Then the Riemannian-Christoffel endomorphism R induces endomorphism r of module $(\Lambda^2)_V(M)$ of vertical 2-forms on M. We find the matrix of the endomorphism as the function system $\{ r_{\beta\gamma\delta\varepsilon} \}$ on the space of fibre bundle $B\mathsf{Q}$. Let $\{ R_{\beta b \gamma c \delta d \varepsilon h} \}$ be components of tensor R on the space of $\mathcal{G}$-structure, Ω be a vertical 2-form on M. Then $R(\Omega)_{\beta b \gamma c} = R_{\beta b \gamma c \delta d \varepsilon h} \Omega^{\delta d \varepsilon h}$. But by (41), $\Omega^{\delta d \varepsilon h} = \omega^{\delta\varepsilon} G^{dh}$, $R(\Omega)_{\beta b \gamma c} = r(\omega)_{\beta\gamma} G_{bc}$ and thus, $r(\omega)_{\beta\gamma} G_{bc} = R_{\beta b \gamma c \delta d \varepsilon h} G^{dh} \omega^{\delta\varepsilon}$, hence, $r(\omega)_{\beta\gamma} = \frac{1}{n} G^{bc} G^{dh} R_{\beta b \gamma c \delta d \varepsilon h} \omega^{\delta\varepsilon}$. Thus,

$$r_{\beta\gamma\delta\varepsilon} = \frac{1}{n} G^{bc} G^{dh} R_{\beta b \gamma c \delta d \varepsilon h}. \tag{56}$$

In view of symmetry properties of tensor R and (56) we get

Proposition 15. *Tensor r has the following symmetry properties:*

$$1)\, r_{\beta\gamma\delta\varepsilon} = -r_{\gamma\beta\delta\varepsilon}; \quad 2)\, r_{\beta\gamma\delta\varepsilon} = -r_{\beta\gamma\varepsilon\delta}; \quad 3)\, r_{\beta\gamma\delta\varepsilon} = r_{\delta\varepsilon\beta\gamma}.$$

Proof. Taking into account equality (56), we have:

$$1)\, r_{\beta\gamma\delta\varepsilon} = \frac{1}{n} R_{\beta b \gamma c \delta d \varepsilon h} G^{bc} G^{dh} = -\frac{1}{n} R_{\gamma c \beta b \delta d \varepsilon h} G^{bc} G^{dh} =$$
$$= -\frac{1}{n} R_{\gamma c \beta b \delta d \varepsilon h} G^{cb} G^{dh} = -r_{\gamma\beta\delta\varepsilon}.$$

Similarly for 2). Finally,

$$3)\, r_{\beta\gamma\delta\varepsilon} = \frac{1}{n} R_{\beta b\gamma c\delta d\varepsilon h}\, G^{bc}\, G^{dh} = \frac{1}{n} R_{\delta d\varepsilon h\beta b\gamma c}\, G^{bc}\, G^{dh} =$$
$$= \frac{1}{n} R_{\delta b\varepsilon c\beta d\gamma h}\, G^{dh}\, G^{bc} = \frac{1}{n} R_{\delta b\varepsilon c\beta d\gamma h}\, G^{bc}\, G^{dh} = r_{\delta\varepsilon\beta\gamma}. \quad \square$$

Summarizing the above we get the following result:

Theorem 31. *Riemann-Christoffel endomorphism of an $\mathcal{H}AQ_\alpha$-manifold of vertical type induces a self-conjugate endomorphism of vertical 2-form module of this manifold.* $\square$

Definition 25. Endomorphism $r : (\Lambda^2)_V(M) \to (\Lambda^2)_V(M)$ will be called *a twistor curvature* (or *t-curvature*) *tensor* of an $\mathcal{H}AQ_\alpha$-manifold M.

The sense of the definition will be made clear in the following chapter. Note that in the case $\dim M = 4$ we have $r = R$, in particular, tensor r components in the frame $(p, e, J_1(e), J_2(e), J_3(e))$; $e \in T_p(M)$, $|e\| = 1$, coincides with tensor R components in this frame. The question naturally arises: When does endomorphism r (in natural reprsentation) preserve the module of self-dual (and anti-self-dual) forms of the manifold ? We know that this module in natural representation corresponds to the Kaehler module of the manifold, and thus, the question can be put in the following way: When does Riemann-Christoffel endomorphism preserve the Kaehler module of an $\mathcal{H}AQ_\alpha$-manifold of vertical type ?

Definition 26. The structural bundle Q of an $\mathcal{H}AQ_\alpha$-manifold M of vertical type will be called *Einsteinian* if on the space of fibre bundle $B\mathsf{Q}$ over this manifold

$$g^{\beta\delta} r_{\beta\gamma\delta\varepsilon} = c g_{\gamma\varepsilon}; \qquad c \in C^\infty(M). \tag{57}$$

It is clear that if $\dim M = 4$, the property of the structural bundle being Einsteinian is equivalent to manifold M being Einsteinian.

Theorem 32. *Riemann-Christoffel endomorphism of an $\mathcal{H}AQ_\alpha$-manifold of vertical type preserves the Kaehler module of the manifold iff the structural bundle of the manifold is Einsteinian.*

Proof. The assertion of the theorem is, evidently, equivalent to the following: endomorphism r preserves in natural representation module $\Lambda^+(M)$ of self-dual forms on $\mathcal{H}AQ_\alpha$-manifold M of vertical type iff on $B\mathsf{Q}$ identities (57) are valid. Let $\omega \in \Lambda^+(M)$. On the space $B(\mathsf{Q})$ this 2-form is characterized by the function

matrix

$$(\omega_{\beta\gamma}) = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & -z & -\alpha y \\ y & z & 0 & \alpha x \\ z & \alpha y & -\alpha x & 0 \end{pmatrix}. \tag{58}$$

Thus, $r(\Lambda^+(M)) \subset \Lambda^+(M) \iff (\forall\, \omega \in \Lambda^+(M) \Longrightarrow r(\omega) \in \Lambda^+(M))$, i.e.

$$\begin{aligned} &1)\ (\alpha r_{01\beta\gamma} + r_{23\beta\gamma})\omega^{\beta\gamma} = 0; \\ &2)\ (\alpha r_{02\beta\gamma} - r_{13\beta\gamma})\omega^{\beta\gamma} = 0; \\ &3)\ (r_{03\beta\gamma} - r_{12\beta\gamma})\omega^{\beta\gamma} = 0. \end{aligned} \tag{59}$$

Further, $\omega^{\beta\gamma} = g^{\beta\delta}\, g^{\gamma\zeta}\, \omega_{\delta\zeta} = \varepsilon(\beta,\gamma)\omega_{\beta\gamma}$, and by (58),

$$(\omega^{\beta\gamma}) = \begin{pmatrix} 0 & \alpha x & \alpha y & -z \\ -\alpha x & 0 & -z & y \\ -\alpha y & z & 0 & -x \\ z & -y & x & 0 \end{pmatrix}.$$

Consequently, correlations (59) have the form

$$\begin{aligned} 1)\ & (\alpha r_{0101} + r_{2301})\alpha x + (\alpha r_{0102} + r_{2302})\alpha y - \\ & -(\alpha r_{0103} + r_{2303})z - (\alpha r_{0112} + r_{2312})z + \\ & +(\alpha r_{0113} + r_{2313})y - (\alpha r_{0113} + r_{2323})x = 0. \end{aligned}$$

In view of the fact that these correlations must equivalently hold with respect to x, y, z, we get:

$$\begin{aligned} & r_{0101} - r_{2323} = 0; \\ & r_{0102} + \alpha r_{2302} + \alpha r_{0113} + r_{2313} = 0; \\ & \alpha r_{0103} + r_{2303} + \alpha r_{0112} + r_{2312} = 0. \end{aligned} \tag{60}$$

$$\begin{aligned} 2)\ & (\alpha r_{0201} - r_{1301})\alpha x + (\alpha r_{0202} - r_{2302})\alpha y - \\ & -(\alpha r_{0203} - r_{1303})z - (\alpha r_{0212} - r_{1312})z + \\ & +(\alpha r_{0213} - r_{1313})y - (\alpha r_{0223} - r_{1323})x = 0; \end{aligned}$$

hence

$$\begin{aligned} & r_{0201} - \alpha r_{1301} - \alpha r_{0223} + r_{1323} = 0; \\ & r_{0202} - r_{1313} = 0; \\ & \alpha r_{0203} - r_{1303} + \alpha r_{0212} - r_{1312} = 0. \end{aligned} \tag{61}$$

$$\begin{aligned} 3)\ & (r_{0301} - r_{1201})\alpha x + (r_{0302} - r_{1302})\alpha y - \\ & -(r_{0303} - r_{1203})z - (r_{0312} - r_{1212})z + \\ & +(r_{0313} - r_{1213})y - (r_{0323} - r_{1223})x = 0; \end{aligned}$$

hence

$$\begin{aligned} & \alpha r_{0301} - \alpha r_{1201} - r_{0323} + r_{1223} = 0; \\ & \alpha r_{0302} - \alpha r_{1202} + r_{0313} - r_{1213} = 0; \\ & r_{0303} - r_{1212} = 0. \end{aligned} \tag{62}$$

Comparing (60_2) and (61_1) we get:

$$r_{0102} + r_{1323} = 0; \qquad r_{2302} + r_{0113} = 0. \tag{63}$$

Similarly, comparing (60_3) and (62_1), (61_3) and (62_2), we get, respectively:

$$\begin{array}{ll} \alpha r_{0112} + r_{0323} = 0; & \alpha r_{0103} + r_{1223} = 0; \\ \alpha r_{0212} - r_{0313} = 0; & \alpha r_{0203} - r_{1213} = 0. \end{array} \tag{64}$$

Equalities (63) and (64) can be rewritted in the form

$$\begin{array}{lll} r_{0102} + r_{3132} = 0; & r_{1013} + r_{2023} = 0; & -\alpha r_{1012} + r_{3032} = 0; \\ r_{0103} - \alpha r_{2123} = 0; & -\alpha r_{2021} + r_{3031} = 0; & r_{0203} - \alpha r_{1213} = 0; \end{array}$$

or $g^{\beta\delta}\, r_{\beta\gamma\delta\varepsilon} = 0 \quad (\gamma \neq \varepsilon)$, i.e. $g^{\beta\delta}\, r_{\beta\gamma\delta\varepsilon} = cg_{\gamma\varepsilon} \quad (\gamma \neq \varepsilon)$. It is asserted that the system of the rest of the equalities

$$r_{0101} = r_{2323}; \quad r_{0202} = r_{1313}; \quad r_{0303} = r_{1212}; \tag{65}$$

is equivalent to equalities $g^{\beta\gamma}\, r_{\beta\gamma\delta\gamma} = cg_{\gamma\gamma}$, i.e.

$$g^{\beta\delta}\, r_{\beta 0\delta 0} = -\alpha g^{\beta\delta}\, r_{\beta 1\delta 1} = -\alpha g^{\beta\delta}\, r_{\beta 2\delta 2} = g^{\beta\delta}\, r_{\beta 3\delta 3},$$

i.e.

$$\begin{array}{l} -\alpha r_{1010} - \alpha r_{2020} + r_{3030} = -\alpha r_{0101} + r_{2121} - \alpha r_{3131} \\ = -\alpha r_{0202} + r_{1212} - \alpha r_{3232} = r_{0303} - \alpha r_{1313} - \alpha r_{2323}. \end{array} \tag{66}$$

Indeed, comparing the first and the forth expressions in (66), we get:

$$r_{1010} + r_{2020} = r_{1313} + r_{2323}.$$

On the other hand, comparing the second and the third expressions in (66), we get:

$$r_{1010} - r_{2020} = -r_{1313} + r_{2323},$$

hence,

$$r_{1010} = r_{2323}, \quad r_{2020} = r_{1313}.$$

In view of the above and comparing the first and the second equalities in (66) we find that $-\alpha r_{2020} + r_{3030} = r_{2121} - \alpha r_{2020}$, i.e. $r_{3030} = r_{1212}$. Evidently, the inverse is also trouth. Combining these results we get that $g^{\beta\delta} r_{\beta\gamma\delta\varepsilon} = cg_{\gamma\varepsilon}$. Inversely, from the above it follows that validity of these conditions is equivalent to (63)-(65) that, as we have seen, are equivalent to (59) by (58), i.e. to module $\Lambda^+(M)$ being invariant with respect to endomorphism r. $\square$

Corollary. *A 4-dimensional oriented pseudo-Riemannian manifold of index 0 or 2 is Einsteinian iff its module of self-dual forms is invariant with respect to Riemann-Chrictoffel endomorphism.* $\square$

The above Corollary shows that the proved Theorem 32 is a broad generalization of the known Atiyah-Hitchin-Singer theorem giving a criterion of 4-dimensional Riemannian manifolds being Einsteinian in terms of self-dual forms [3] since it generalizes this theorem in case of neutral pseudo-Riemannian metric itself.

5 t-conformal-semiflat Manifolds

The great importance of conformal-semiflat, i.e. self-dual or anti-self-dual, manifolds in geometry and theoretical physics is explained by the fact that the canonical almost complex structure on their twistor spaces is integrable that allows to have twistor interpretation of Yang-Mills fields - instantons and anti-instantons respectively, on the manifolds generalizing the classical Ward theorem [3].

In the previous chapter we introduced the notion of twistor curvature tensor (endomorphism) of $\mathcal{H}AQ_\alpha$-manifolds of vertical type. Now consider the trace-free part of the endomorphism that we call *a Weyl endomorphism.* Using it we get a natural generalization of self-dual and anti-self-dual manifolds for the case of $\mathcal{H}AQ_\alpha$-manifolds of arbitrary dimension. We call them *t-conformal-semiflat* $\mathcal{H}AQ_\alpha$-manifolds and show that their twistor curvature tensor is the algebraic curvature tensor. The notion of twistor curvature of $\mathcal{H}AQ_\alpha$-manifold is introduced and $\mathcal{H}AQ_\alpha$-manifolds of constant twistor curvature are studied. It is proved that anti-self-dual $\mathcal{H}AQ_\alpha$-manifolds, as well as generalized quaternionic-Kaehler manifolds are $\mathcal{H}AQ_\alpha$-manifolds of constant twistor curvature.

5.1 Self-dual and anti-self-dual $\mathcal{H}AQ_\alpha$-manifolds

Let M be an $\mathcal{H}AQ_\alpha$-manifold of vertical type, Q be its structural bundle, $r : (\Lambda^2)_V(M) \to (\Lambda^2)_V(M)$ be a twistor curvature tensor of manifold M. Construct tensor $ric \in (\mathcal{T}_2^0)_V(M)$ with components $ric_{\beta\varepsilon} = g^{\gamma\delta}\, r_{\beta\gamma\delta\varepsilon}$ that we call *Ricci twistor tensor*, or *Ricci t-curvature tensor.* By Theorem 32 a Riemann-Christoffel endomorphism of an $\mathcal{H}AQ_\alpha$-manifold of vertical type preserves the Kaehler module of the manifold iff its Ricci twistor tensor is proportional to tensor g of fibre metric. The trace $\kappa = g^{\beta\gamma}\, ric_{\beta\gamma}$ of tensor ric will be called *a t-scalar curvature* of manifold M. By means of tensor ric we construct endomorphism W of module $(\Lambda^2)_V(M)$ with components

$$W_{\beta\gamma\delta\varepsilon} = r_{\beta\gamma\delta\varepsilon} + \frac{1}{2}(ric_{\beta\delta}g_{\gamma\varepsilon} + ric_{\gamma\varepsilon}g_{\beta\delta} - ric_{\beta\varepsilon}g_{\gamma\delta} - ric_{\gamma\delta}g_{\beta\varepsilon}) + \frac{\kappa}{6}(g_{\gamma\delta}g_{\beta\varepsilon} - g_{\beta\delta}\, g_{\gamma\varepsilon})$$

generalizing the classical Weyl tensor of conformal curvature of a 4-dimensional pseudo-Riemannian manifold. We call it *a Weyl endomorphism.* Evidently, it has all symmetry properties of a twistor curvature tensor found in Proposition 15. Besides, it is evident that $g^{\gamma\delta}W_{\beta\gamma\delta\varepsilon} = 0$. Similarly to the proof of Theorem 32 for endomorphism r we get for Weyl endomorphism:

Theorem 33. *Submodules of self-dual and anti-self-dual forms on an $\mathcal{H}AQ_\alpha$-manifold of vertical type are invariant with respect to the action of Weyl endomorphism.* □

Thus, $W = W^+ + W^-$, where $W^\pm$ are endomorphism W narrowings onto submodules $\Lambda^\pm(M)$, respectively.

Definition 27. An $\mathcal{H}AQ_\alpha$-manifold M of vertical type will be called *self-dual* (resp., *anti-self-dual*) if $W^- = 0$ (resp., $W^+ = 0$). An self-dual or anti-self-dual $\mathcal{H}AQ_\alpha$-manifold will be called *t-conformal-semiflat.*

¿From the above we see that the notions generalize the corresponding notions of 4-dimensional Riemannian geometry that have become classical [5].

Introduce *self-duality index* ξ of a t-conformal-semiflat $\mathcal{H}AQ_\alpha$-manifold equal to 1 for an self-dual and -1 for an anti-self-dual manifold.

Let M be an $\mathcal{H}AQ_\alpha$-manifold, $\omega \in \Lambda^\pm(M)$, i.e. $*(\omega) = \pm\omega$. As we have seen in the proof of Theorem 23, on the space of fibre bundle $B\mathsf{Q}$, where $(g_{\beta\gamma}) = diag(1, -\alpha, -\alpha, 1)$, these equalities will assume the form $\omega_{\hat\beta\hat\gamma} = \pm\varepsilon(\beta,\gamma)\,\omega_{\beta\gamma}$, respectively, where $\varepsilon(\beta,\gamma) = g_{\beta\beta}\, g_{\gamma\gamma}$; $\beta, \gamma = 0, 1, 2, 3, 4$; $(\hat\beta, \hat\gamma)$ is the pair complementing the pair (β, γ) up to even permutation of indices (0,1,2,3). It means that

$$(\omega_{\beta\gamma}) = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & \mp z & \mp y \\ y & \pm z & 0 & \pm\alpha x \\ z & \pm\alpha y & \mp\alpha x & 0 \end{pmatrix};\ (\omega^{\beta\gamma}) = \begin{pmatrix} 0 & \alpha x & \alpha y & -z \\ -\alpha x & 0 & \mp z & \pm y \\ -\alpha y & \pm z & 0 & \mp x \\ z & \mp y & \pm x & 0 \end{pmatrix}. \tag{67}$$

Now let M be a t-conformal-semiflat $\mathcal{H}AQ_\alpha$-manifold. If M is self-dual, then $W(\omega) = 0 \quad (\omega \in \Lambda^-(M))$. If M is anti-self-dual, then $W(\omega) = 0 \quad (\omega \in L^+(M))$. By (67) these conditions will be written in the form $(\alpha W_{\beta\gamma01} + \xi W_{\beta\gamma23})x + (\alpha W_{\beta\gamma02} - \xi W_{\beta\gamma13})y + (\xi W_{\beta\gamma12} - W_{\beta\gamma03})z = 0$. Since these correlations must equivalently hold with respect to x, y and z, we get that they are equivalent to the conditions

$$W_{\beta\gamma01} = -\alpha\xi W_{\beta\gamma23}, \qquad W_{\beta\gamma02} = \alpha\xi W_{\beta\gamma13}, \qquad W_{\beta\gamma03} = \xi W_{\beta\gamma12}.$$

Write the correlations in view of symmetry properties of tensor W, that are similar to symmetry properties of tensor r, found in Proposition 15:

$$\begin{aligned}
&W_{1002} = -W_{1332} = -\alpha\xi W_{0113} = \alpha\xi W_{0223};\\
&W_{1003} = \alpha W_{1223} = -\xi W_{0112} = -\alpha\xi W_{0332};\\
&W_{2003} = \alpha W_{2113} = \xi W_{0221} = \alpha\xi W_{0331};\\
&W_{0101} = W_{2323} = -\alpha\xi W_{0123};\\
&W_{0202} = W_{1313} = -\alpha\xi W_{0231};\\
&W_{0303} = W_{1212} = \xi W_{0312}.
\end{aligned}$$

By definition of tensor W the equalities will be rewritten, respectively, in the

form:

$$\begin{array}{l}
1)\ r_{1002}+\alpha\xi r_{0113}=\frac{1}{2}(ric_{12}-\xi\, ric_{03});\\
2)\ r_{1332}+\alpha\xi r_{0223}=\frac{1}{2}(ric_{12}-\xi\, ric_{03});\\
3)\ r_{1003}+\xi r_{0112}=\frac{1}{2}(ric_{13}-\alpha\xi\, ric_{02});\\
4)\ r_{1223}+\xi r_{0332}=\frac{1}{2}(\xi\, ric_{02}-\alpha\, ric_{13});\\
5)\ r_{2003}-\xi r_{0221}=\frac{1}{2}(ric_{23}+\alpha\xi\, ric_{01});\\
6)\ r_{2113}-\xi r_{0331}=\frac{1}{2}(\xi\, ric_{01}-\alpha\, ric_{23});\\
7)\ r_{0101}+\alpha\xi r_{0123}=\frac{1}{2}(\alpha\, ric_{00}-ric_{11})-\frac{1}{6}\alpha\kappa;\\
8)\ r_{2323}+\alpha\xi r_{0123}=\frac{1}{2}(\alpha\, ric_{33}-ric_{22})-\frac{1}{6}\alpha\kappa;\\
9)\ r_{0202}+\alpha\xi r_{0231}=\frac{1}{2}(\alpha\, ric_{00}-ric_{22})-\frac{1}{6}\alpha\kappa;\\
10)\ r_{1313}+\alpha\xi r_{0231}=\frac{1}{2}(\alpha\, ric_{33}-ric_{11})-\frac{1}{6}\alpha\kappa;\\
11)\ r_{0303}-\xi r_{0312}=-\frac{1}{2}(ric_{00}+ric_{33})+\frac{1}{6}\kappa;\\
12)\ r_{1212}-\xi r_{0312}=\frac{\alpha}{2}(ric_{11}+ric_{22})+\frac{1}{6}\kappa.
\end{array}\qquad (68)$$

Besides, by definition of tensor ric,

$$\begin{array}{ll}
r_{1002}+r_{1332}=ric_{12}; & r_{1003}-\alpha r_{1223}=ric_{13};\\
r_{2003}-\alpha r_{2113}=ric_{23}; & -\alpha r_{0112}+r_{0332}=ric_{02};\\
r_{0113}+r_{0223}=-\alpha ric_{03}; & -\alpha r_{0221}+r_{0331}=ric_{01}.
\end{array}\qquad (69)$$

¿From the above immediately follows

Theorem 34. *A twistor curvature tensor of t-conformal-semiflat an $\mathcal{H}AQ_\alpha$-manifold is an algebraic curvature tensor, tensor W being its trace-free part, i.e. the classical Weyl tensor with respect to tensor r.*

Proof. By (68_7), (68_9) and (68_{11}) we have: $r_{0123}+r_{0231}+r_{0312}=-\alpha\xi(r_{0101}+r_{0202})+\xi r_{0303}+\frac{1}{2}\xi(ric_{00}\alpha\, ric_{11}+ric_{00}-\alpha\, ric_{22}+ric_{00}+ric_{33})-\frac{1}{2}\xi\kappa=\xi(\alpha\, r_{0110}+\alpha\, r_{0220}-r_{0330})+\frac{1}{2}\xi(ric_{00}-\alpha\, ric_{11}-\alpha\, ric_{22}+ric_{33})+\xi\, ric_{00}\xi\kappa=-\xi\, ric_{00}+\frac{1}{2}\xi\kappa+\xi\, ric_{00}-\frac{1}{2}\xi\kappa=0$. By Proposition 15 we get that $r_{\beta[\gamma\delta\varepsilon]}=0$, and thus, r is an algebraic curvature tensor of module $(\Lambda^2)_V(M)$. For such tensor, as is well-known [5], its trace-free part is uniquely defined by the formula that gives tensor W. $\square$

Consider some examples of t-conformal-semiflat $\mathcal{H}AQ_\alpha$-manifolds.

Definition 28. An $\mathcal{H}AQ_\alpha$-structure that is at the same time a πAQ_α-structure, whose structural endomorphisms are parallel in Riemannian connection is called *a generalized hyper-Kaehler structure.*

Theorem 35. *Any generalized hyper-Kaehler manifold of dimension greater than 4 is t-conformal-semiflat.*

Proof. As we will see below (Theorem 39), any generalized hyper-Kaehler manifold M is Ricci-flat. Besides, it, evidently, is a generalized quaternionic-Kaehler manifold, and by (55) and Theorem 28, $R(\Omega)=0\quad(\Omega\in(\Lambda^2)_V(M))$. By definition of twistor curvature tensor $r=0$. Thus, $ric=0$, and $W=0$. In particular, manifold M is t-conformal-semiflat. $\square$

Corollary. *All $\mathcal{H}AQ_\alpha$-manifolds of dimension greater than 4, mentioned in example 2, section2.2, are t-conformal-semiflat.* $\Box$

5.2 Some properties of t-conformal-semiflat manifolds

Theorem36. *A twistor curvature endomorphism of an self-dual (resp., anti-self-dual) $\mathcal{H}AQ_\alpha$-manifold M transfers all anti-self-dual (resp., self-dual) forms on M into self-dual (resp., anti-self-dual) iff M is a manifold of zero t-scalar curvature.*

Proof. Endomorphism r has the given property iff

$$1)\ \alpha r(\omega)_{01} = \mp r(\omega)_{23}; \quad 2)\ \alpha r(\omega)_{02} = \pm r(\omega)_{13}; \quad 3)\ r(\omega)_{03} = \pm r(\omega)_{12};$$

$\omega \in \Lambda^{\mp}(M)$. Writing the equalities in view of (68) and (69) we get that they are equivalent to the condition $ric_{00} - \alpha\, ric_{11} - \alpha\, ric_{22} + ric_{33} = 0$, or $\kappa = tr(ric) = g^{\beta\gamma}\, ric_{\beta\gamma} = 0$. $\Box$

Theorem 37. *A twistor curvature endomorphism of an self-dual (resp., anti-self-dual)$\mathcal{H}AQ_\alpha$-manifold M transfers all vertical 2-forms on M into self-dual (resp., anti-self-dual) iff M is manifold of zero Ricci t-curvature.*

Proof. Let endomorphism r transfers any vertical 2-form on M into self-dual (resp, anti-self-dual). In particular, any self-dual form is transferred into self-dual (in the second case it is in view of endomorphism r being self-conjugated). By Theorem 32 ric is a scalar endomorphism, and by Theorem 36 its trace is equal to zero. Thus, $ric = 0$. Inversely, if $ric = 0$, then by the same Theorems, $r(\Lambda^{\pm}(M)) \subset \Lambda^{\pm}(M)$, $\quad r(\Lambda^{\mp}(M)) \subset \Lambda^{\pm}(M)$, and hence, $r(\Lambda_V(M)) \subset \Lambda^{\pm}(M)$. $\Box$

Corollary. *Riemann-Christoffel endomorphism R of a 4-dimensional conformal-semiflat pseudo-Riemannian manifold of index 0 or 2 transfers all anti-self-dual forms on the manifold into self-dual (or self-dual into anti-self-dual, depending on the choice of orientation of manifold) iff the scalar curvature of the manifold is zero. Here, R transfers any 2-forms on manifold into self-dual (or anti-self-dual) iff the manifold is Ricci-flat.* $\Box$

Let M be a self-dual (resp., anti-self-dual) $\mathcal{H}AQ_\alpha$-manifold, $p \in M$, $\omega \in \Lambda^{\mp}(M)$ is a non-isotropic eigenvector of its endomorphism r of twistor curvature in the point. Without loss of generality the vector norm can be considered equal to either $\sqrt{-\alpha}$ or 1. In the former case we choose orthonormalized basis $\{\,\mathrm{id}, J_1, J_2, J_3\,\}$ of the space $\mathsf{Q}_p(M)$ so that ω coincides with the pseudofunda-

mental (resp., fundamental) form of α-quaternion J_1. By (67), in this basis

$$(\omega^{\beta\gamma}) = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi \\ 0 & 0 & -\xi & 0 \end{pmatrix}.$$

Thus equality $r(\omega) = \lambda\omega$ will be rewritten in view of (68) and (69) in the form

$$\begin{aligned}
&1)\ \tfrac{2}{3}(\alpha\, ric_{00} - ric_{11}) - \tfrac{1}{3}(\alpha\, ric_{33} - ric_{22}) = \alpha\lambda;\\
&2)\ ric_{12} + \xi\, ric_{03} = 0;\\
&3)\ ric_{13} + \alpha\xi\, ric_{02} = 0;\\
&4)\ -\tfrac{1}{3}(\alpha\, ric_{00} - ric_{11}) + \tfrac{2}{3}(\alpha\, ric_{33} - ric_{22}) = \alpha\lambda.
\end{aligned}$$

Note that the first and third of the equalities can be further rewritten in the form, respectively:

$$\begin{aligned}
2(\alpha\, ric_{00} - ric_{11}) - (\alpha\, ric_{33} - ric_{22}) &= 3\alpha\lambda;\\
-(\alpha\, ric_{00} - ric_{11}) + 2(\alpha\, ric_{33} - ric_{22}) &= 3\alpha\lambda;
\end{aligned}$$

hence,

$$\begin{aligned}
&1)\ \alpha ric_{00} - ric_{11} = 3\alpha\lambda; && 2)\ \alpha ric_{33} - ric_{22} = 3\alpha\lambda;\\
&3)\ ric_{12} + \xi\, ric_{03} = 0; && 4)\ ric_{13} + \alpha\xi\, ric_{02} = 0;
\end{aligned}$$

or

$$\begin{aligned}
&1)\ ric_0^0 + ric_1^1 = 3\lambda; && 2)\ ric_3^3 + ric_2^2 = 3\lambda;\\
&3)\ ric_1^2 - \alpha\xi\, ric_0^3 = 0; && 4)\ ric_1^3 - \xi\, ric_0^2 = 0.
\end{aligned}$$

In particular, $\kappa = \operatorname{tr} ric = 6\lambda$. Now, let ω have a positive norm. Assume $\|\omega\| = 1$. Choose orthonormalized basis $\{\,\mathrm{id}, J_1, J_2, J_3\,\}$ of the space $\mathsf{Q}_p(M)$ so that ω coincides with pseudofundamental (resp., fundamental) form of α-quaternion J_3. By (67), in the basis

$$(\omega^{\beta\gamma}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \xi & 0 \\ 0 & -\xi & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this case equality $r(\omega) = \lambda\omega$ by (68) and (69) will be rewritten in the form

$$\begin{aligned}
&1)\ \tfrac{2}{3}(ric_{00} + ric_{33}) + \tfrac{1}{3}\alpha(ric_{11} + ric_{22}) = \lambda;\\
&2)\ \tfrac{1}{3}(ric_{00} + ric_{33}) + \tfrac{2}{3}\alpha(ric_{11} + ric_{22}) = -\lambda;\\
&3)\ ric_{13} - \alpha\xi\, ric_{02} = 0;\\
&4)\ ric_{23} + \alpha\xi\, ric_{01} = 0.
\end{aligned}$$

The first two equalities can be rewritten in the form, respectively:

$$\begin{aligned}
2(ric_{00} + ric_{33}) + \alpha(ric_{11} + ric_{22}) &= 3\lambda;\\
-(ric_{00} + ric_{33}) - 2\alpha(ric_{11} + ric_{22}) &= 3\lambda;
\end{aligned}$$

hence,

$$\begin{array}{ll} 1)\ ric_{00}+ric_{33}=3\lambda; & 2)\ ric_{11}+ric_{22}=-3\alpha\lambda; \\ 3)\ ric_{13}-\alpha\xi ric_{02}=0; & 4)\ ric_{23}+\alpha\xi ric_{01}=0; \end{array}$$

or

$$\begin{array}{ll} 1)\ ric_0^0+ric_3^3=3\lambda; & 2)\ ric_1^1+ric_2^2=3\lambda; \\ 3)\ ric_3^1+\xi\, ric_2^0=0; & 4)\ ric_3^2-\xi\, ric_1^0=0. \end{array}$$

In particular, $\kappa = \operatorname{tr}(ric) = 6\lambda$. Thus we have proved

Theorem 38. *Eigenvalue of non-isotropic anti-self-dual (or self-dual) form ω, which is eigenvector of twistor curvature endomorphism of an self-dual (resp., anti-self-dual) $\mathcal{H}AQ_\alpha$-manifold is equal to $\frac{1}{6}\kappa$, where κ is a t-scalar curvature of the manifold.* □

By Theorem 36 we get

Corollary 1. *If an self-dual (resp., anti-self-dual) $\mathcal{H}AQ_\alpha$-manifpld admits at any its point a non-isotropic anti-self-dual (resp., self-dual) form belowing to the kernel of the twistor curvature endomorphism, then the t-scalar curvature of the man fold at this point is zero, and thus, the image of any anti-self-dual (resp., self-dual) form at the given point is an self-dual (resp., anti-self-dual) form in view of the endomorphism.* □

By Theorem 37 we also get

Corollary 2. *Let M be an self-dual (resp., anti-self-dual) $\mathcal{H}AQ_\alpha$-manifold. If its twistor curvature endomorphism at any point of the manifold vanishes at least one non-isotropic anti-self-dual (resp., self-dual) form, them M is a manifold of zero t-scalar curvature. If the endomorphism turns to zero any anti-self-dual (resp., self-dual) form on M, then M is a manifold of Ricci-zero t-curvature.* □

Corollary 3. *Let M be a 4-dimensional self-dual (resp., anti-self-dual) pseudo-Riemannian manifold of index 0 or 2. If its Riemann-Ghristoffel endomorphism at every point of the manifold vanishes at least one non-isotropic anti-self-dual (resp., self-dual) form, then M is a zero scalar curvature manifold. If the endomorphism vanishes any anti-self-dual (resp., self-dual) form on M, then M is a Ricci-flat manifold.* □

5.3 Twistor curvature of $\mathcal{H}AQ_\alpha$-manifold

Let M be an $\mathcal{H}AQ_\alpha$-manifold of vertical type, $J \in \{\mathsf{T}\}$ be a twistor on M, $\Omega(X,Y) = \langle X, JY\rangle$ be its Kaehler form.

Definition 29. Function

$$k(J) = \frac{R(\Omega,\Omega)}{\|\Omega\|^2} = \frac{\langle R(\Omega),\Omega\rangle}{\|\Omega\|^2}$$

is called *a twistor curvature* (or *t-curvature*) *of manifold M in the direction of twistor J.*

Since by Theorem 26 Ω is a vertical 2-form on M, its components on the space of G-structure (by (41) and Theorem 23) will have the form $\Omega_{\beta b\gamma c} = \omega_{\beta\gamma}\, G_{bc}$, where $\{\, \omega_{\beta\gamma} \,\}$ are components on the space of fibre bundle $B\mathsf{Q}$ of self-dual form ω which is the fundamental form of α-quaternion J. Thus,

$$\begin{aligned}\langle R(\Omega),\Omega\rangle &= -\tfrac{1}{2}R_{\beta b\gamma c\delta d\zeta h}\Omega^{\delta d\zeta h}\Omega^{\beta b\gamma c}\\ &= -\tfrac{1}{2}R_{\beta b\gamma c\delta d\zeta h}\omega^{\delta\zeta}\,\omega^{\beta\gamma}G^{dh}G^{bc} = \tfrac{n}{2}r_{\beta\gamma\delta\zeta}\omega^{\beta\gamma}\omega^{\delta\zeta} = n(r(\omega),\omega),\end{aligned}$$

and since $\|\Omega\|^2 = n\omega^2$, we have:

$$k(J) = \frac{(r(\omega),\omega)}{\|\omega\|^2},$$

and thus, tensor r completely defines a twistor curvature on M, that explains the name of the tensor.

Definition 30. An $\mathcal{H}AQ_\alpha$-manifold M is called *a manifold of pointwise constant twistor curvature c* if

$$\forall\ J \in \{\,\mathsf{T}\,\} \Longrightarrow \langle R(\Omega),\Omega\rangle = c\|\Omega\|^2; \qquad c \in C^\infty\,(M).$$

If here $c = const$, M is called *a manifold of globally constant twistor curvature.*

Example 1. Let M^{4n} be an $\mathcal{H}AQ_\alpha$-manifold of constant curvature c. Then

$$\begin{aligned}\langle R(\Omega),\Omega\rangle &= -\tfrac{1}{2}R_{\beta b\gamma c\delta d\zeta h}\,\Omega^{\beta b\gamma c}\,\Omega^{\delta d\zeta h}\\ &= \tfrac{c}{2}(G_{\beta b\delta d}\,G_{\gamma c\zeta h} - G_{\gamma c\delta d}\,G_{\beta b\zeta h})\Omega^{\beta b\gamma c}\,\Omega^{\delta d\zeta h}\\ &= \tfrac{c}{2}(g_{\beta\delta}\,g_{\gamma\zeta}\,G_{bd}\,G_{ch} - g_{\gamma\delta}\,g_{\beta\zeta}\,G_{cd}\,G_{bh})\omega^{\beta\gamma}\,G^{bc}\,\omega^{\delta\zeta}\,G^{dh}\\ &= c\omega_{\delta\zeta}\omega^{\delta\zeta}\,G_{cd}\,G^{cd} = 2c\|\Omega\|^2,\end{aligned}$$

i.e. M is a manifold of globally constant twistor curvature.

Example 2. Let M^{4n} be a generalized quaternionic-Kaehler manifold, $\Omega \in \mathcal{K}(M)$, I be its corresponding twistor. Then at an arbitrary point $p \in M$, by (47) and (52) that are, evidently true in any manifold dimension,

$$\begin{aligned}\langle R(\Omega),\Omega\rangle &= \tfrac{1}{2}\langle R(\Omega)_{ij}\Omega^{ij}\rangle = \tfrac{1}{2}\textstyle\sum_{i,j=1}^{4n}\langle R(\Omega)e_i,e_j\rangle\Omega(e_i,e_j)\\ &= -\tfrac{\alpha n}{2(n+2)}\textstyle\sum_{i,j=1}^{4n} Ric(e_i,Ie_j)\langle e_i,Ie_j\rangle\\ &= -\tfrac{\alpha n}{2(n+2)}\textstyle\sum_{i,j=1}^{4n} Ric(e_i,e_j)\langle e_i,e_j\rangle\\ &= -\tfrac{\alpha n}{2(n+2)}\textstyle\sum_{i=1}^{4n} Ric(e_i,e_i)\|e_i\|^2\\ &= -\tfrac{\alpha ns}{2(n+2)} = -\tfrac{\alpha s}{4(n+2)}\|\Omega\|^2,\end{aligned}$$

because

$$\begin{aligned}\|\Omega\|^2 &= \tfrac{1}{2}\Omega_{ij}\,\Omega^{ij} = \tfrac{1}{2}\textstyle\sum_{i,j=1}^{4n}\langle e_i, e_j\rangle^2 = \tfrac{1}{2}\sum_{i,j=1}^{4n}\langle e_i, I^2 e_j\rangle^2\\ &= \tfrac{1}{2}\textstyle\sum_{i,j=1}^{4n}\langle e_i, e_j\rangle^2 = \tfrac{1}{2}\sum_{i=1}^{4n}\|e_i\|^2 = 2n\end{aligned}$$

Here $\{\, e_1, \dots, e_{4n} \,\}$ is orthonormalized basis adapted to twistor I. Thus, any generalized quaternionic-Kaehler manifold is a manifold of constant twistor curvature $c = -\frac{\alpha s}{4(n+2)}$, wnere s is the scalar curvature of the manifold. Moreover, if $\dim M > 4$, by Theorem 29, M is an Einsteinian manifold, and thus, the manifold of globally constant twistor curvature $c = -\frac{\alpha n\varepsilon}{n+2}$, where ε is Einstein constant. In particular, all quaternionic-Kaehler manifolds given in examples 2 and 3 of section 2.2, are manifolds of constant twistor curvature.

Example 3. Let $\{\, I, J \,\}$ is a generalized hyper-Kaehler structure, i.e. πAQ_α-structure generated by twistors I and J that are parallel in Riemannian connection, on pseudo-Riemannian manifold M. Let $J_1 = I$, $J_2 = J$, $J_3 = I \circ J$. Then $\nabla_X (J_\beta) = 0; \quad X \in \mathsf{X}(M) \quad \beta = 1, 2, 3$. Thus, $[R(X,Y), \mathcal{J}] = 0; \quad X, Y \in \mathsf{X}(M), \quad \mathcal{J} \in \{\, \mathsf{T} \,\}$. In particular, $R(\mathcal{I}) \circ \mathcal{J} = \mathcal{J} \circ R(\mathcal{I}); \quad \mathcal{I}, \mathcal{J} \in \{\, \mathsf{T} \,\}$, i.e. $R(\mathcal{I})$ belongs to centre of α-quaternion algebra, hence, $R(\mathcal{I}) = c\,\mathrm{id}$. In view of the fact that $R(\Omega)$ is skew-symmetric tensor, where Ω is a Kaehler form of twistor $\mathcal{I}$, we get $R(\Omega) = 0$, and thus, $\langle R(\Omega), \Omega\rangle = 0 \quad (\Omega \in \mathcal{K}(M))$, i.e. M is a manifold of zero twistor curvature.

Moreover, since the generalized hyper-Kaehler structure is, evidently, generalized quaternionic-Kaehler, by the above example we get that if $\dim M > 4$, then $\varepsilon = 0$, i.e. M is a Ricci-flat manifold. If $\dim M = 4$, then by above example $s = 0$, and since $R(\Omega) = 0 \quad (\Omega \in \mathcal{K}(M))$, M is an Einsteinian manifold (by the Corollary of Theorem 32), and thus, $Ric = 0$, i.e. M is Ricci-flat, too. Thus, we have proved

Theorem 39. *Any generalized hyper-Kaehler manifold is Ricci-flat.* $\square$

The above theorem generalizes the known Berger result of Ricci curvature of hyper-Kaehler manifolds [6].

Let M be an arbitrary $\mathcal{H}AQ_\alpha$-manifold of vertical type of pointwise constant twistor curvature c. From the above it follows that this is equivalent to

$$\langle r(\omega), \omega\rangle = c\langle \omega, \omega\rangle; \qquad \omega \in \Lambda^+(M).$$

Polarizing the equality and in view of the endomorphism r being self-conjugated, we find that $\langle r(\omega), \vartheta\rangle = c\langle \omega, \vartheta\rangle; \quad \omega, \vartheta \in \Lambda^+(M)$. Thus, $r(\omega) = c\omega + \varphi; \quad \varphi \in \Lambda^-(M)$. Evidently, the inverse is also true. Thus, we get

Theorem 40. *An $\mathcal{H}AQ_\alpha$-manifold M of vertical type is a manifold of pointwise constant twistor curvature c iff $(r - c\,\mathrm{id})\omega \in \Lambda^-(M); \omega \in \Lambda^+(M)$.* $\square$

Let M be an $\mathcal{H}AQ_\alpha$-manifold, $\omega \in (\Lambda^2)_V(M)$, $\{\,\omega_{\beta\gamma}\}$ are components of tensor ω on the space $B\mathbb{Q}$. We have: $\omega = \omega^+ + \omega^-$; $\quad \omega^\pm \in \Lambda^\pm(M)$. By (67) we get:

$$(\omega^+_{\beta\gamma}) = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & z & \alpha y \\ -y & -z & 0 & -\alpha x \\ -z & -\alpha y & \alpha x & 0 \end{pmatrix}; \quad (\omega^-_{\beta\gamma}) = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & -\alpha b \\ -b & c & 0 & \alpha a \\ -c & \alpha b & -\alpha a & 0 \end{pmatrix};$$

Hence, $\omega_{01} = x + a$; $\omega_{02} = y + b$; $\omega_{03} = z + c$; $\omega_{23} = -\alpha x + \alpha a$; $\omega_{13} = \alpha y - \alpha b$; $\omega_{12} = z - c$. Thus,

$$x = \frac{1}{2}(\omega_{01} - \alpha\omega_{23}); \qquad y = \frac{1}{2}(\omega_{02} + \alpha\omega_{13}); \qquad z = \frac{1}{2}(\omega_{03} + \omega_{12}). \tag{70}$$

These correlations define projection $\pi : (\Lambda^2)_V(M) \to \Lambda^+(M)$ along $\Lambda^-(M)$.

Now let M be an anti-self-dual $\mathcal{H}AQ_\alpha$-manifold. Compute $\pi \circ r(\omega)$; $\quad \omega \in \Lambda^+(M)$. We have: $r(\omega)_{\beta\gamma} = -r_{\beta\gamma\delta\zeta}\,\omega^{\delta\zeta}$. By (68) and (69) it is easy to compute that

$$r(\omega)_{01} - \alpha r(\omega)_{23} = \frac{1}{3}\kappa x; \quad r(\omega)_{02} + \alpha r(\omega)_{13} = \frac{1}{3}\kappa y; \quad r(\omega)_{03} + r(\omega)_{12} = \frac{1}{3}\kappa z.$$

In view of (70), this means that

$$x(\pi \circ r(\omega)) = \frac{1}{6}\kappa x(\omega); \qquad y(\pi \circ r(\omega)) = \frac{1}{6}\kappa y(\omega); \qquad z(\pi \circ r(\omega)) = \frac{1}{6}\kappa z(\omega).$$

(Here x, y and z are coordinate functions). Thus, $\pi \circ r(\omega) = \frac{1}{6}\kappa\omega$ and $\pi \circ (r - \frac{1}{6}\kappa\,\mathrm{id})(\omega) = 0$, i.e. $(r - \frac{1}{6}\kappa\,\mathrm{id})\omega \in \Lambda^-(M)$. By Theorem 40 we get:

Theorem 41. *An anti-self-dual $\mathcal{H}AQ_\alpha$-manifold is manifold of pointwise constant twistor curvature $c = \frac{1}{6}\kappa$, where κ is a t-scalar curvature of the manifold.* $\square$

Corollary 1. *A 4-dimensional anti-self-dual pseudo-Riemannian manifold of index 0 or 2 is a manifold of pointwise constant twistor curvature $c = \frac{1}{6}s$, where s is a scalar curvature of the manifold.* $\square$

Corollary 2. *A 4-dimensional anti-self-dual pseudo-Riemannian manifold whose self-dual form module is invariant with respect to the Riemann-Christoffel endomorphism is a manifold of globally constant twistor curvature.*

Proof. This immediately follows from the manifold being Einsteinian (by Theorem 32), in particular, the manifold of constant scalar curvature. $\square$

6 Self-dual Geometry of 4-dimensional Kaehler Manifolds

The relation of geometry of conformal-semiflat manifolds to geometry of Einsteinian manifolds [5], as well as to twistor geometry [7] is of great interest to the researchers. For example, the well-known Penrose-Atiyah-Hitchin-Singer theorem [3] asserts that the canonical almost complex structure of the twistor space of a 4-dimensional oriented Riemannian manifold (M, g) is integrable iff (M, g) is conformal-semiflat. Hitchin showed that if (M, g) is also a compact Einsteinian manifold of positive scalar curvature, then it is isometric to S^4 or $\mathbf{C}P^2$ with standard metrics [8]. This result is also closely connected with twistor geometry: Hitchin proved that a 4-dimensional oriented compact Riemannian manifold (M, g) has a Kaehler twistor space iff (M, g) is conformally equivalent to S^4 or $\mathbf{C}P^2$ with their standard conformal structures [9]. B.-Y.Chen [10] and (independently and by another method) J.P.Bourguignon [11] and A.Derdzinski [12] got a classification of self-dual compact Kaehler manifolds, and M.Itoh [13] got a classification of self-dual Kaehler-Einsteinian manifolds. Besides, M.Itoh gave a complete characteristics of compact anti-self-dual Kaehler manifolds [13].

In the present chapter we received a complete classification of self-dual generalised Kaehler manifolds (both of classical and non-exceptional Kaehler manifolods of hyperbolic type) of constant scalar curvature. It is proved that a generalized Kaehler manifold is anti-self-dual iff its scalar curvature is equal to zero. The results essentially generalized the above mentioned results of N.Hitchin, J.P.Bourguignon, A.Derdzinski, B.-Y.Chen and M.Itoh.

6.1 Generalized almost Hermitian structures

Definition 31 [14]. We call *a generalized* (in the narrow sense) *almost Hermitian* (or, $A\mathcal{H}_\alpha-$it structure) on a manifold M a pair $\{\, g = \langle \cdot, \cdot \rangle, J \,\}$ of tensor field on M, where g is a pseudo-Riemannian metric, J is an endomorphism of module $\mathsf{X}(M)$, such that $J^2 = \alpha\,\mathrm{id}; \quad \alpha = \pm 1$, and called *a structural operator*, or *structural endomorphism.* Here, $\langle JX, JY \rangle = -\alpha \langle X, Y \rangle; \quad X, Y \in \mathsf{X}(M)$. If $\alpha = -1$, the $A\mathcal{H}_\alpha$-structure is called *an almost Hermitian structure of classical type*, if $\alpha = 1$, it is called *an almost Hermitian structure of hyperbolic type* or, in other terms, *an almost para-Hermitian srtucture.* In case $\alpha = 1$, as is well-known, metric g is neutral, i.e. it has a zero signature.

A 2-form $\Omega(X, Y) = \langle X, JY \rangle; \quad X, Y \in \mathsf{X}(M)$ called *a fundamental structure form* is associated to every $A\mathcal{H}_\alpha$-structure. The form is non-degenerate and, thus, its exterior square generates the orientation of manifold M.

Definition 32. We call *a canonical orientation* on $A\mathcal{H}_\alpha$-manifold (M, g, J) an orientation coherent to the form $-\alpha\Omega \wedge \Omega$.

We know [14] that giving an $A\mathcal{H}_\alpha$-structure (M,J,g) on M equivalent to giving G-structure G on M with structural group $G = U(2,\mathbf{K}_\alpha)$, where $\mathbf{K}_\alpha$ is the field $\mathbf{C}$ of complex numbers in the classical case or the ring $\mathbf{D}$ of double numbers in the hyperbolic case. The space elements of the G-structure are called *frames adapted to the $A\mathcal{H}_\alpha$-structure*, or *A-frames* [14]. Recall the construction of A-frames. Let $(p, e_0, e_1, Je_0, Je_1)$ be an orthonormalized frame on manifold M adapted to structural endomorphism J, $\|e_0\|^2 = 1$, $\|e_1\|^2 = -\alpha$. Then its corresponding A-frame $(p, \varepsilon_0, \varepsilon_1, e_{\hat{0}}, e_{\hat{1}})$ is defined as a frame in the space $T_p(M)\otimes\mathbf{K}_\alpha$ with vectors $\varepsilon_a = \frac{1}{\sqrt{2}}(e_a + iJ^3 e_a)$, $\varepsilon_{\hat{a}} = \frac{1}{\sqrt{2}}(e_a - iJ^3 e_a)$; $\quad a, b, c \ldots = 0, 1$; i is the imaginary unit of the ring $\mathbf{K}_a$. Note that $\overline{\varepsilon}_a = \varepsilon_{\hat{a}}$, $J\varepsilon_a = i\varepsilon_a$, $J\varepsilon_{\hat{a}} = -i\varepsilon_a$, where $X \to \overline{X}$ is the operator of natural involution (of conjugation) in $\mathbf{K}_\alpha$-modul $T_p(M)\otimes\mathbf{K}_\alpha$.

Compute the matrix of metric tensor g components on the space G. In the frame $(p, e_0, e_1, Je_0, Je_1)$ we have by definition: $(g_{ik}) = diag(1, -\alpha, -\alpha, 1)$. Thus, in the A-frame $(p, \varepsilon_0, \varepsilon_1, \varepsilon_{\hat{0}}, \varepsilon_{\hat{1}})$ we have: $g_{0\hat{0}} = \langle \varepsilon_0, \varepsilon_{\hat{0}}\rangle = \frac{1}{2}\langle e_0 + i\alpha J e_0, e_0 - i\alpha J e_0\rangle = \frac{1}{2}\{\langle e_0, e_0\rangle + \langle e_0, e_0\rangle\} = 1 = g_{\hat{0}0}$. Similarly, $g_{1\hat{1}} = g_{\hat{1}1} = -\alpha$. The rest of the metric tensor components are equal to zero. Thus,

$$(g_{ij}) = (g^{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\alpha \\ 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \end{pmatrix} \tag{71}$$

$(i, j = 0, 1, \hat{0}, \hat{1})$. Evidently, $\det(g_{ij}) = 1$. On the other hand, compute fundamental form of the structure $\Omega(X,Y) = \langle X, JY\rangle$: $\Omega_{0\hat{0}} = \langle \varepsilon_0, J\varepsilon_{\hat{0}}\rangle = -i = -\Omega_{\hat{0}0}$. Similarly, $\Omega_{1\hat{1}} = -\Omega_{\hat{1}1} = i\alpha$. The rest of the components of the form Ω are equal zero. Thus, $\Omega = \Omega_{ij}\,\omega^i \wedge \omega^j = -2i\omega^0 \wedge \omega^{\hat{0}} + 2i\alpha\omega^1 \wedge \omega^{\hat{1}}$. Hence, in particular,

$$\Omega \wedge \Omega = -4\alpha\sqrt{\det g}\,\omega^0 \wedge \omega^1 \wedge \omega^{\hat{0}} \wedge \omega^{\hat{1}}$$

and, thus, A-frames are positively oriented with respect to the canonical orientation in the classical case. In particular, $\eta_{01\hat{0}\hat{1}} = -\alpha$. In view of this equality and by (71) we can specify the conditions of self-duality and anti-self-duality of 2-forms on an $A\mathcal{H}_\alpha$-manifold, given by correlations (7), on the space of G-structure:

Lemma 4. *Let (M, g, J) be a 4-dimensional generalized almost Hermitian manifold, $\omega \in \Lambda^2(M)$. In this case*

1. $\omega \in \Lambda^+(M)$ iff on the space G

$$(\omega_{jk}) = \begin{pmatrix} 0 & x+iz & iy & 0 \\ -x-iz & 0 & 0 & -\alpha y \\ -iy & 0 & 0 & x-iy \\ 0 & i\alpha y & -x+iz & 0 \end{pmatrix};$$

2. $\omega \in \Lambda^-(M)$ *iff on the space* G

$$(\omega_{jk}) = \begin{pmatrix} 0 & 0 & iy & x+iz \\ 0 & 0 & -x+iz & i\alpha y \\ -iy & x-iz & 0 & 0 \\ -x-iz & -i\alpha y & 0 & 0 \end{pmatrix}.$$

Proof. Let $\omega \in \Lambda^{\pm}(M)$. By (7), (71) and the given remarks about the components of volume form η_g we have:

$$\begin{aligned}
&1)\ \pm\omega_{01} = \tfrac{1}{2}\eta_{01ij}\, g^{ik}\, g^{jm}\, \omega_{km} = \eta_{01\hat{0}\hat{1}}\, g^{\hat{0}0}\, g^{\hat{1}1}\, \omega_{01} = \omega_{01};\\
&2)\ \pm\omega_{0\hat{0}} = \tfrac{1}{2}\eta_{0\hat{0}ij}\, g^{ik}\, g^{jm}\, \omega_{km} = \eta_{0\hat{0}1\hat{1}}\, g^{1\hat{1}}\, g^{\hat{1}1}\, \omega_{\hat{1}1} = -\alpha\omega_{1\hat{1}};\\
&3)\ \pm\omega_{0\hat{1}} = \tfrac{1}{2}\eta_{0\hat{1}ij}\, g^{ik}\, g^{jm}\, \omega_{km} = \eta_{0\hat{1}\hat{0}1}\, g^{\hat{0}0}\, g^{1\hat{1}}\, \omega_{0\hat{1}} = -\omega_{0\hat{1}};
\end{aligned}$$

Similarly, $\pm\omega_{1\hat{0}} = -\omega_{1\hat{0}}$; $\pm\omega_{\hat{0}\hat{1}} = \omega_{\hat{0}\hat{1}}$. Besides, since ω is a real tensor, $\omega_{\hat{i}\hat{j}} = \overline{\omega}_{ij}$ $(\hat{a} = a)$. In particular, $\overline{\omega}_{a\hat{a}} = -\omega_{a\hat{a}}$. All the correlations are, evidently, equivalent to the assertion of Lemma 4. □

Definition 33. An $A\mathcal{H}_\alpha$-srtucture (g, J) is called *a generalized Kaehler* (or, $\mathcal{K}_\alpha-$) *structure*, if structural endomorphism J is parallel in Riemannian connection ∇ of the manifold.

In case $\alpha = -1$ this notion coincides with the classical notion of Kaehler structure, in case $\alpha = 1$ it coincides with the notion of Kaehler structure of hyperbolic type, or in other terms, para-Kaehler structure [15]. Manifolds carring a para-Kaehler structure are also known as *Rashevskii fibre bundles*. The name is explained by the existence on such manifolds, first studied by P.K.Rashevskii of two isotropic totally integrable totally geodesic distributions - eigendistributions of the structural endomorphism complementary to each other [16].

Let (M, g, J) be a $\mathcal{K}_\alpha$-manifold. We know [14] that the first group of Cartan equations of such manifold on the space of G-structure G has the form:

$$1)\ d\omega^a = \omega^a_b \wedge \omega^b; \qquad 2)\ d\omega_a = -\omega^b_a \wedge \omega_b; \tag{72}$$

where $\omega_a = \varepsilon(a)\omega^{\hat{a}} = \varepsilon(a)\overline{\omega}^a$, $\overline{\omega}^a_b = -\varepsilon(a,b)\omega^b_a$; $\varepsilon(a) = g_{aa}$, $\varepsilon(a,b) = g_{aa}\, g_{bb}$. By exterior differentiation of (72) and in view of basis forms being linearly independent we get the second group of structural equations of $\mathcal{K}_\alpha$-structure:

$$d\omega^a_b = \omega^a_c \wedge \omega^c_b + A^{ad}_{bc}\, \omega^c \wedge \omega_d; \tag{73}$$

where $\{\, A^{ad}_{bc} \,\}$ is the function system on space G symmetric by subscripts or superscripts and satisfying the system of differential equations

$$dA^{ad}_{bc} + A^{ad}_{hc}\, \omega^h_b + A^{ad}_{bh}\, \omega^h_c - A^{hd}_{bc}\, \omega^a_h - A^{ah}_{bc}\, \omega^d_h = A^{ad}_{bch}\, \omega^h + A^{adh}_{bc}\, \omega_h; \tag{74}$$

where $\{ A^{ad}_{bch}, A^{adh}_{bc} \}$ is the function system on space G symmetric by any pair of subscrips or superscripts and serving as components of covariant differential in the Riemannian connection of tensor A defined, in view of (74), by the function system $\{ A^{ad}_{bc} \}$ and called *the tensor of holomorphic sectional curvature* [17]. Comparing (72) and (73) with structural equations of the pseudo-Riemannian structure we have

$$d\omega^i = \omega^i_j \wedge \omega^j, \qquad d\omega^i_j = \omega^i_k \wedge \omega^k_j + \frac{1}{2} R^i_{jkm} \omega^k \wedge \omega^m,$$

where $\{ R^i_{jkm} \}$ are components of Riemann-Christoffel tensor R. It is easy to compute the explicit components on space G (i.e. in A-frame):

$$R_{\hat{a}bc\hat{d}} = R_{b\hat{a}\hat{d}c} = -R_{\hat{a}b\hat{d}c} = -R_{b\hat{a}c\hat{d}} = \varepsilon(a,d) A^{ad}_{bc}. \tag{75}$$

The rest of the tensor components are equal to zero. Now we can compute the components of the trace-free part of tensor R, the Weyl tensor W of conformal curvature. We know that tensors R and W are connected by the correlation ($\dim M = 4$):

$$\begin{aligned} W_{ijkm} &= R_{ijkm} + \tfrac{1}{2}(r_{ik}\, g_{jm} + r_{jm}\, g_{ik} - r_{im}\, g_{jk} - \\ &= -r_{jk}\, g_{im}) + \tfrac{s}{6}(g_{im}\, g_{jk} - g_{ik}\, g_{jm}), \end{aligned} \tag{76}$$

where $r_{ik} = g^{jm} R_{ijmk}$ are Ricci tensor components, $s = g^{ik} r_{ik}$ is the scalar curvature of the manifold. In view of (71) and (75) it is easy to compute that in A-frame $g_{ab} = g_{\hat{a}\hat{b}} = 0$, $g_{a\hat{b}} = g_{\hat{b}a} = \varepsilon(b)\delta^b_a$; $r_{ab} = r_{\hat{a}\hat{b}} = 0$; $r_{a\hat{b}} = r_{\hat{b}a} = -\varepsilon(b) A^{cb}_{ca}$. Thus, (76) in A-frame will have the form: $W_{ab\hat{c}\hat{d}} = \frac{1}{2}(r_{a\hat{c}}\, g_{b\hat{d}} + r_{b\hat{d}}\, g_{a\hat{c}} - r_{a\hat{d}}\, g_{b\hat{c}} - r_{b\hat{c}}\, g_{a\hat{d}}) + \frac{s}{6}(g_{a\hat{d}}\, g_{b\hat{c}} - g_{a\hat{c}}\, g_{b\hat{d}})$, or

$$W_{ab}{}^{cd} = \frac{1}{2}(r^c_a\, \delta^d_b + r^d_b\, \delta^c_a - r^d_a\, \delta^c_b - r^c_b\, \delta^d_a) + \frac{s}{6}(\delta^d_a\, \delta^c_b - \delta^c_a\, \delta^d_b). \tag{77}$$

Similarly, $W_{a\hat{b}c\hat{d}} = R_{a\hat{b}c\hat{d}} - \frac{1}{2}(r_{a\hat{d}}\, g_{\hat{b}c} + r_{\hat{b}c}\, g_{a\hat{d}}) + \frac{s}{6} g_{\hat{b}c}\, g_{a\hat{d}}$, or

$$W_a{}^b{}_c{}^d = -A^{bd}_{ac} - \frac{1}{2}(r^d_a\, \delta^b_c + r^b_c\, \delta^d_a) + \frac{s}{6}\delta^b_c\, \delta^d_a, \tag{78}$$

as well as correlations defined by symmetry properties of tensor W (similar to symmetry properties of Riemann-Christoffel tensor).

6.2 4-dimensional self-dual $\mathcal{K}_\alpha$-manifolds

Let M be a 4-dimensional oriented pseudo-Riemannian manifold of index 0 or 2, W be its Weyl tensor of conformal curvature. Consider W as an endomorphism of module $\Lambda^2(M)$. It is known [5] that submodules $\Lambda^{\pm}(M)$ are invariant with respect to the endomorphism, and thus, $W = W^+ + W^-$, where $W^{\pm}$ are restrictions of endomorphism W onto submodules $\Lambda^{\pm}(M)$, respectively.

Definition 34 ([3],[10]). A 4-dimensional oriented pseudo-Riemannian manifold of index 0 (or 2) is called *self-dual* (respectively, *anti-self-dual*) if $W^- = 0$ (respectively, $W^+ = 0$) for it. An self-dual or anti-self-dual manifold is called *conformal-semiflat.*

Example. Let (M, g, J) be a 4-dimensional generalized complex space form, i.e. a generalized Kaehler manifold of constant holomorphic sectional curvature. It is easy to check that a $\mathcal{K}_\alpha$-manifold is a generalized complex space form iff

$$A^{ad}_{bc} = c\,\tilde{\delta}^{ad}_{bc},$$

where $c = const$, $\tilde{\delta}^{ad}_{bc} = \delta^a_b\,\delta^d_c + \delta^a_c\,\delta^d_b$. In this case $r^a_b = \varepsilon(a) r_{\hat{a}b} = -A^{ca}_{cb} = -c\,\tilde{\delta}^{ca}_{cb} = -3c\,\delta^a_b$, $s = 2r_{a\hat{b}}\,g^{a\hat{b}} = 2r^a_a = -12c$, and by (77) and (78) $W_{ab}{}^{cd} = -c\,\delta^{cd}_{ab}$, where $\delta^{cd}_{ab} = \delta^c_a\,\delta^d_b - \delta^c_b\,\delta^d_a$; $W_a{}^b{}_c{}^d = \varepsilon(b,d) R_{a\hat{b}c\hat{d}} + 3c\,\delta^d_a\,\delta^b_c - 2c\,\delta^b_c\,\delta^d_a = -A^{bd}_{ac} + c\delta^b_c\,\delta^d_a = -c\,\tilde{\delta}^{bd}_{ac} + c\delta^b_c\,\delta^d_a = -c\delta^b_a\,\delta^d_c$. Thus, if $\vartheta \in \Lambda^-(M)$, then by Lemma 4

$$W(\vartheta)_{ab} = -W_{ab}{}^{cd}\,\vartheta_{cd} = 0; \qquad W(\vartheta)^a{}_b = -W^a{}_b{}^c{}_d\,\vartheta^d{}_c = -c\,\delta^a_b\,\vartheta^c{}_c = 0,$$

i.e. $W(\omega) = 0$. Thus, we get

Theorem 42. *A 4-dimensional generalized complex space form is an self-dual manifold.* $\square$

Now let M be an arbitrary 4-dimensional self-dual generalized Kaehler manifold, $\vartheta \in \Lambda^-(M)$. Then $W(\vartheta) = 0$, and by (78) $W(\vartheta)^b{}_a = B^{bd}_{ac}\,\vartheta^c{}_d = 0$, where

$$B^{bd}_{ac} = A^{bd}_{ac} + \frac{1}{2}(r^b_c\,\delta^d_a + r^d_a\,\delta^b_c) - \frac{1}{6}\delta^b_c\,\delta^d_a.$$

In view of Lemma 4 this equality will be written in the form: $(B^{b1}_{a1} - B^{b0}_{a0})iy + (\alpha B^{b0}_{a1} + B^{b1}_{a0})x + (\alpha B^{b0}_{a1} - B^{b1}_{a0})iz = 0$. In view of $x, y, z \in C^\infty(M)$ being arbitrary, we have:

$$B^{b1}_{a1} - B^{b0}_{a0} = 0; \qquad B^{b0}_{a1} = B^{b1}_{a0} = 0; \qquad (a, b = 0, 1). \tag{79}$$

On the other hand,

$$B^{bc}_{ac} = A^{bc}_{ac} + \frac{1}{2}(r^b_a + r^b_a) - \frac{s}{6}\,\delta^b_a = -\frac{s}{6}d^b_a. \tag{80}$$

Comparing (79) and (80) we get $B^{b0}_{a0} = B^{b1}_{a1} = -\frac{s}{12}\,\delta^b_a$. Thus, $B^{bd}_{ac} = -\frac{s}{12}\,\delta^b_a\,\delta^d_c$. In other word,

$$A^{bd}_{ac} = -\frac{1}{2}(r^b_c\,\delta^d_a + r^d_a\,\delta^b_c) + \frac{s}{6}\,\delta^b_c\,\delta^d_a - \frac{s}{12}\,\delta^b_a\,\delta^d_c. \tag{81}$$

Alternating the equqlity by indices b and d we get:

$$r^b_c\,\delta^d_a + r^d_a\,\delta^b_c - r^b_a\,\delta^d_c - r^d_c\,\delta^b_a = \frac{s}{2}\,\delta^{bd}_{ac}. \tag{82}$$

It is immediately checked that the above is equivalent to the condition $r^a_a = \frac{1}{2}s$ that holds automatically in view of Ricci tensor being real. On the other hand, symmetrize (81) by indices b and d:

$$A^{bd}_{ac} = -\frac{1}{4}(r^b_c\,\delta^d_a + r^d_a\,\delta^b_c + r^b_a\,\delta^d_c + r^d_c\,\delta^b_a) + \frac{s}{24}\,\tilde{\delta}^{bd}_{ac}. \tag{83}$$

In view of the above remark the equality is equivalent to (81). Now, note that (83) is equivalent to the following:

$$A^{bd}_{ac} = t^{(b}_{(a}\,\delta^{d)}_{c)}; \tag{84}$$

where t is a tensor; the brackets mean symmetrization by the enclosed indices. Indeed, let (84) be true, i.e. $4A^{bd}_{ac} = t^b_a\,\delta^d_c + t^d_c\,\delta^b_a + t^b_c\,\delta^d_a + t^d_a\,\delta^b_c$. Contracting the equality by indices d and c we get:

$$-4r^b_a = 4t^b_a + tr(t)\,\delta^b_a, \text{ and thus,} \qquad t^b_a = -r^b_a - \frac{1}{4}tr(t)\,\delta^b_a.$$

Contracting the equality, we get $tr(t) = -\frac{1}{3}s$, and thus

$$t^b_a = -r^b_a + \frac{s}{12}\,\delta^b_a. \tag{85}$$

Inversely, substituting (85) into (84) we get (83).

Let (84) and the equivalent (83) hold. Substituting (82) and (83) into (77) and (78), respectively, we get:

$$W_{ab}{}^{cd} = \frac{s}{6}\,\delta^{cd}_{ab}; \qquad W_a{}^b{}_c{}^d = \frac{s}{12}\,\delta^b_a\,\delta^d_c,$$

that, in view of Lemma 4, implies $W(\omega) = 0 \quad (\omega \in \Lambda^-(M))$, i.e. M is an self-dual manifold. Thus, we have proved

Theorem 43. *A 4-dimensional generalized Kaehler manifold is self-dual iff $A^{bd}_{ac} = t^{(b}_{(a}\delta^{d)}_{c)}$, where t is a tensor, and, necessarily, $t^b_a = -r^b_a + \frac{1}{12}s\,\delta^b_a$.* $\square$

Definition 35. A generalized Kaehler manifold is called *non-exceptional* if its Ricci endomorphism has at least one non-isotropic eigenvector at every point of the manifold.

Example. Any Kaehler manifold of classical type (with definite metric) is non-exceptional since, in view of its Ricci endomorphism r being self-conjugate, it has at every its point an orthonormalized basis of tangent space at this point

consisting of eigenvectors of the endomorphism. Moreover, since on the space of G-structure G components r_{ab} and $r_{\hat{a}\hat{b}}$ of Ricci endomorphism are equal to zero,the endomorphism commutes with the structural endomorphism J, and thus, the mentioned basis can be choose adapted to J, i.e. having the form (e_0, e_1, Je_0, Je_1). A-basis generated by this basis will be the eigenbasis of endomorphism r, as well as t, and eigenvalues of the endomorphosms will be real.

Moreover, endomorphisms r and t of any non-exceptional $\mathcal{K}_\alpha$-manifold have similar properties. Indeed, in view of the fact that for a $\mathcal{K}_\alpha$-manifold M there holds the equality $r \circ J = J \circ r$, endomorphism r preserves eigensubmodules of endomorphism J, in particular, it admits narrowing on eigensubmodules $D_J^{\pm i}$ of the endomorphism with eigenvalues $\pm i$, respectively. Let e be a non-isotropic eigenvector of endomorphism r at a point of M. Without loss of generality it can be considered unitary. Then Je is also a non-isotropic eigenvector of this endomorphism and it corresponds to the same eigenvalue $\tilde{\lambda}$. Thus, $\varepsilon_0 = \frac{1}{\sqrt{2}}(e + iJ^3 e)$ is an eigenvector of endomorphism r with the same (real) eigenvalue. Comlementing the pair $\{\, \varepsilon_0, \varepsilon_{\hat{0}} \,\}$, $\quad \varepsilon_{\hat{0}} = \frac{1}{\sqrt{2}}(e - iJ^3 e)$ up to A-basis $\{\, \varepsilon_0, \varepsilon_1, \varepsilon_{\hat{0}}, \varepsilon_{\hat{1}} \,\}$ we get that in basis $\{\, \varepsilon_0, \varepsilon_1 \,\}$ of submodule D_J^i at this point endomorphism r has matrix

$$(r) = \begin{pmatrix} \tilde{\lambda} & a \\ 0 & \tilde{\mu} \end{pmatrix}; \qquad a, \tilde{\mu} \in \mathbf{K}_\alpha.$$

On the other hand, from (73) it follows that $\overline{A}^{ad}_{bc} = A^{bc}_{ad}$, and thus $\overline{r}^a_b = r^b_a$. Therefore, $a = 0$, $\tilde{\mu} \in \mathbf{R}$, i.e. endomorphism $r|D_J^i$ is diagonalized in basis $\{\, \varepsilon_0, \varepsilon_1 \,\}$. Moreover, since in view of endomorphism r being real $\overline{r}^a_b = r^{\hat{b}}_{\hat{a}}$, $r|D_J^{-i}$ is diagonalized in basis $\{\, \varepsilon_{\hat{0}}, \varepsilon_{\hat{1}} \,\}$ and given by the same matrix. Thus, endomorphism r (as well as t) is diagonalized both in A-basis $\{\, \varepsilon_0, \varepsilon_1, \varepsilon_{\hat{0}}, \varepsilon_{\hat{1}} \,\}$ and in the corresponding orthonormalized basis $\{\, e_0, e_1, Je_0, Je_1 \,\}$.

Now, let (M, g, J) be a non-exceptional a $\mathcal{K}_\alpha$-manifold. By the above we can consider subbundle G_1 of G-srtucture G, consisting of A-frames, where

$$(t^b_a) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}; \qquad \lambda = \tilde{\lambda} - \frac{s}{12},\ \mu = \tilde{\mu} - \frac{s}{12}. \tag{86}$$

Note that by Theorem 43, $\lambda + \mu = -\frac{1}{3}s$. By (74) components of tensor t on the space G satisfy the differential equations

$$dt^a_b + t^a_c\,\omega^c_b - t^c_b\,\omega^a_c = t^a_{bc}\,\omega^c + t^{ac}_b\,\omega_c,$$

where $t^a_{bc} = A^{ha}_{hbc} - \frac{2}{3}A^{hr}_{hrc}\,\delta^a_b$, $t^{ac}_b = \varepsilon(c)\,A^{hac}_{hb} - \frac{2}{3}A^{hrc}_{hr}\,\delta^a_b$. In particular, on the space of G_1-structure, where $t^a_b = \lambda_b\,\delta^a_b$, $\lambda_0 = \lambda$, $\lambda_1 = \mu$, we have:

$$\begin{array}{lll} 1)\ d\lambda = \lambda_c\,\omega^c + \lambda^c\,\omega_c; & \lambda_c = t^0_{0c},\ \lambda^c = t^{0c}_0; \\ 2)\ d\mu = \mu_c\,\omega^c + \mu^c\,\omega_c; & \mu_c = t^1_{1c},\ \mu^c = t^{1c}_1; \\ 3)\ (\lambda - \mu)\omega^0_1 = F_c\,\omega^c + F^c\,\omega_c; & F_c = t^0_{1c},\ F^c = t^{0c}_1; \\ 4)\ (\mu - \lambda)\omega^1_0 = G_c\,\omega^c + G^c\,\omega_c; & G_c = t^1_{0c},\ G^c = t^{1c}_0. \end{array} \tag{87}$$

We write structural equations of a generalized Kaehler structure on the space G_1:

$$\begin{array}{l}
1)\ d\omega^0 = \omega^0_0 \wedge \omega^0 + \omega^0_1 \wedge \omega^1;\\
2)\ d\omega^1 = \omega^1_0 \wedge \omega^0 + \omega^1_1 \wedge \omega^1;\\
3)\ d\omega_0 = -\omega^0_0 \wedge \omega_0 - \omega^1_0 \wedge \omega_1;\\
4)\ d\omega_1 = -\omega^0_1 \wedge \omega_0 - \omega^1_1 \wedge \omega_1;\\
5)\ d\omega^0_0 = \omega^0_1 \wedge \omega^1_0 + \lambda\omega^0 \wedge \omega_0 + \frac{1}{4}(\lambda+\mu)\omega^1 \wedge \omega_1;\\
6)\ d\omega^1_0 = \omega^1_0 \wedge \omega^0_0 + \omega^1_1 \wedge \omega^1_0 + \frac{1}{4}(\lambda+\mu)\omega^1 \wedge \omega_0;\\
7)\ d\omega^0_1 = \omega^0_0 \wedge \omega^0_1 + \omega^0_1 \wedge \omega^1_1 + \frac{1}{4}(\lambda+\mu)\omega^0 \wedge \omega_1;\\
8)\ d\omega^1_1 = \omega^1_0 \wedge \omega^0_1 + \frac{1}{4}(\lambda+\mu)\omega^0 \wedge \omega_0 + \mu\omega^1 \wedge \omega_1.
\end{array} \tag{88}$$

By exterior differentiation of (88_5) and by (88) we have:

$$\{-\frac{1}{2}(\lambda-\mu)\omega^1_0 - \lambda^1\,\omega_0 - \frac{1}{4}(\lambda_0+\mu_0)\omega^1\,\} \wedge \omega^0 \wedge \omega_1 +$$
$$+\{\,\frac{1}{2}(\lambda-\mu)\omega^0_1 - \lambda_1\,\omega^0 + \frac{1}{4}(\lambda^0+\mu^0)\omega_1\,\} \wedge \omega^1 \wedge \omega_0 = 0.$$

In view of (87) and linear independence of basic forms we get $G^0 = 2\lambda^1$; $G_1 = \frac{1}{2}(\lambda_0+\mu_0)$; $F_0 = 2\lambda_1$; $F^1 = \frac{1}{2}(\lambda^0+\mu^0)$, and thus,

$$(\mu-\lambda)\omega^1_0 = G_0\,\omega^0 + \frac{1}{2}(\lambda_0+\mu_0)\omega^1 + 2\lambda^1\omega_0 + G^1\omega_1;$$
$$(\mu-\lambda)\omega^0_1 = -2\lambda_1\,\omega^0 - F_1\,\omega^1 - F^0\,\omega_0 - \frac{1}{2}(\lambda^0+\mu^0)\omega_1. \tag{89}$$

By exterior differentiation of (88_6) and by (88) we have:

$$\begin{array}{l}
(\mu-\lambda)\omega^1_0 \wedge \omega^0 \wedge \omega_0 - (\lambda-\mu)\omega^1_0 \wedge \omega^1 \wedge \omega_1 +\\
+\frac{1}{2}(\lambda_0+\mu_0)\omega^0 \wedge \omega^1 \wedge \omega_0 + \frac{1}{2}(\lambda^1-\mu^1)\omega^1 \wedge \omega_0 \wedge \omega_1 = 0.
\end{array}$$

In view of (89) and linear independence of basic forms we get $G_0 = 0$; $G^1 = 0$; $\mu^1 = 3\lambda^1$, and thus,

$$\begin{array}{l}
(\mu-\lambda)\omega^1_0 = \frac{1}{2}(\lambda_0+\mu_0)\omega^1 + 2\lambda^1\,\omega_0;\\
(\mu-\lambda)\omega^0_1 = -2\lambda_1\,\omega^0 - \frac{1}{2}(\lambda^0+\mu^0)\omega_1.
\end{array} \tag{90}$$

Similarly, by exterior differentiation of (88_8) and by (88) we have:

$$\begin{array}{l}
\frac{1}{2}(\lambda-\mu)\omega^0_1 \wedge \omega^1 \wedge \omega_0 - \frac{1}{2}(\lambda-\mu)\omega^1_0 \wedge \omega^0 \wedge \omega_1 + \frac{1}{4}(\lambda_1+\mu_1)\omega^1 \wedge \omega^0 \wedge \omega_0 +\\
+\frac{1}{4}(\lambda^1+\mu^1)\omega_1 \wedge \omega^0 \wedge \omega_0 + \mu_0\,\omega^0 \wedge \omega^1 \wedge \omega_1 + \mu^0\omega_0 \wedge \omega^1 \wedge \omega_1 = 0;
\end{array}$$

hence, by (90) $\lambda_0 = 3\mu_0$, $\lambda^0 = 3\mu^0$, $\mu_1 = 3\lambda_1$; $\mu^1 = 3\lambda^1$. Thus, $d\lambda = 3\mu_0\,\omega^0 + \lambda_1\,\omega^1 + 3\mu^0\,\omega_0 + \lambda^1\,\omega_1$; $\quad d\mu = \mu_0\,\omega^0 + 3\lambda_1\,\omega^1 + \mu^0\,\omega_0 + 3\lambda^1\,\omega_1$. In particular,

$$ds = -3(d\lambda + d\mu) = -12(\mu_0\,\omega^0 + \lambda_1\,\omega^1 + \mu^0\,\omega_0 + \lambda^1\,\omega_1).$$

Let M be a manifold of constant scalar curvature. Then it follow that $\mu_0 = \mu^0 = \lambda_1 = \lambda^1 = 0$, i.e. $\lambda = const$, $\mu = const$. Here (90) assumes the form:

$$1)\ (\mu - \lambda)\omega_0^1 = 0; \qquad 2)\ (\mu - \lambda)\omega_1^0 = 0.$$

Consider the possible cases.

I. $\lambda \neq \mu$. Then $\omega_0^1 = \omega_1^0 = 0$. By ($88_6$) in this case $\lambda = -\mu$, and (88) assumes the form:

$$\begin{array}{ll} d\omega^0 = \omega_0^0 \wedge \omega^0; & d\omega^1 = \omega_1^1 \wedge \omega^1; \\ d\omega_0 = -\omega_0^0 \wedge \omega_0; & d\omega_1 = -\omega_1^1 \wedge \omega_1; \\ d\omega_0^0 = \lambda\, \omega^0 \wedge \omega_0; & d\omega_1^1 = -\lambda\, \omega^1 \wedge \omega_1. \end{array}$$

In this case M is locally holomorphically isometric to the product of 2-dimensional manifold S_λ^2 of constant curvature λ by 2-dimensional manifold $S_{-\lambda}^2$ of constant curvature $(-\lambda)$. Note that such manifold is conformally flat [5].

II. $\lambda = \mu$. Then $t_b^a = \lambda\, \delta_b^a$, $A_{bc}^{ad} = \frac{1}{4}\lambda\, \tilde{\delta}_{bc}^{ad}$, i.e. M is a generalized complex space form. In this case, by [17], M is locally holomorphically isometric to one of the following manifolds:

1. Complex plane $\mathbf{C}^2$;
2. Complex projective plane $\mathbf{C}P^2$;
3. Complex hyperbolic plane $\mathbf{C}H^2$;
4. Double Euclidean plane $\mathbf{R}^2 \bowtie \mathbf{R}^2$;
5. The space of null-pairs $\mathbf{R}P^2 \odot \mathbf{R}P^2 = GL(3, \mathbf{R})/GL(2, \mathbf{R}) \times GL(1, \mathbf{R})$ of real projective plane;

equipped by a canonical Kaehler structure of classical (in cases 1,2,3) or hyperbolic (in cases 4 and 5) types. We get the following result:

Theorem 44. *An self-dual nonexeptional generalized Kaehler manifold of constant scalar curvature is locally holomorphically isometric to one of the following manifolds: 1)* $\mathbf{C}^2$*; 2)* $\mathbf{C}P^2$*; 3)* $\mathbf{C}H^2$*; 4)* $S_\lambda^2 \times S_{-\lambda}^2$*; 5)* $\mathbf{R}^2 \bowtie \mathbf{R}^2$*; 6)* $\mathbf{R}P^2 \odot \mathbf{R}P^2$*; equipped by a canonical Kaehler structure of classical or, respectively, hyperbolic type.* $\square$

Corollary. *Any self-dual compact Kaehler manifold of classical type is holomorphically isometrically covered with one of the following manifolds: 1)* $\mathbf{C}^2$*; 2)* $\mathbf{C}P^2$*; 3)* $\mathbf{C}H^2$*; 4)* $S_\lambda^2 \times S_{-\lambda}^2$*; equipped by a canonical Kaehler structure.*

Proof. It was shown by A.Derdzinski [12], that any such manifold is locally symmetric, and thus, is a manifold of constant scalar curvature. To complete the proof we apply Theorem 44 and note that all manifolds in the above Corollary are simple-connected. $\square$

Remark. Self-dual generalized Kaehler manifolds of non-constant scalar curvature do not admit such finite classification. A.Derdzinski showed [18] that on $\mathbf{C}^2$ there exists a 2-parametric family of self-dual Kaehler metrics of non-constant scalar curvature.

6.3 Self-duality and Bochner tensor

The geometry of Kaehler manifolds is a complex analog of Riemannian geometry: such important notions of Riemannian geometry as sectional curvature, space forms and many others have its complex counterpart which is of quite non-trivial sense in geometry of Kaehlerian and, more generally, almost Hermitian manifolds. One of such notions is Weyl's conformal curvature tensor which is the basic object of study of conformal geometry. In 1949 S.Bochner introduced the complex analog of the tensor for Kaehlerian manifolds [19]. The tensor introduced by S.Bochner and called after his name, posseses all symmetry properties of Riemann-Chrystoffel tensor and has meaning for arbitrary almost Hermitian manifolds [20]. However, it has quite a complicated structure, and in spite of a considerable number of works devoted to its studying we have comparatively little information about its geometry.

In the present section the Bochner curvature tensor of generalized Kaehlerian manifolds is introduced. The structure of the tensor is studied. The main result of this section states that Bochner curvature tensor of four-dimensional generalized Kaehler manifold vanishes iff the manifold is self-dual. Hence it is proved that Bochner-flat generalized Kaehler manifolds (i.e. generalized Kaehler-Bochner manifolds) are natural generalization of self-dual generalized Kaehler manifolds.

Definition 36. Let M be $2n$-dimensional generalized Kaehler manifold. The tensor B of type (3,1) on M defined by the equality

$$\begin{aligned} B(X,Y)Z = R(X,Y)Z &+ \langle Y,Z\rangle L(X) - \langle X,Z\rangle L(Y) + \langle L(Y),Z\rangle X- \\ &-\langle L(X),Z\rangle Y - \langle J^3(Y),Z\rangle L\circ J(X) + \langle J^3(X),Z\rangle L\circ J(Y)- \\ &-\langle L\circ J^3(Y),Z\rangle J(X) + \langle L\circ J^3(X),Z\rangle J(Y)+ \\ &+2\langle L\circ J^3(X),Y\rangle J(Z) + 2\langle J^3(X),Y\rangle L\circ J(Z); \end{aligned}$$

where $L = \frac{1}{2(n+2)}\, r + \frac{s}{8(n+1)(n+2)}\,\mathrm{id}$, is called *Bochner tensor of manifold M.*

Let as compute components of Bochner tensor in A-frame:

$$\begin{aligned}
B_{\hat{a}bc\hat{d}} &= \langle B(\varepsilon_c, \varepsilon_{\hat{d}}), \varepsilon_b, \varepsilon_{\hat{a}}\rangle = \langle R(\varepsilon_c, \varepsilon_{\hat{d}}), \varepsilon_b, \varepsilon_{\hat{a}}\rangle + \langle \varepsilon_{\hat{d}}, \varepsilon_b\rangle\langle L(\varepsilon_c), \varepsilon_{\hat{a}}\rangle - \\
&\quad - \langle \varepsilon_c, \varepsilon_b\rangle\langle L(\varepsilon_{\hat{d}}), \varepsilon_{\hat{a}}\rangle + \langle L(\varepsilon_{\hat{d}}), \varepsilon_b\rangle\langle \varepsilon_c, \varepsilon_{\hat{a}}\rangle - \langle L(\varepsilon_c), \varepsilon_b\rangle\langle \varepsilon_{\hat{d}}, \varepsilon_{\hat{a}}\rangle - \\
&\quad - laJ^3(\varepsilon_{\hat{d}}), \varepsilon_b\rangle\langle L \circ J(\varepsilon_c), \varepsilon_{\hat{a}}\rangle + \langle J^3(\varepsilon_c), \varepsilon_b\rangle\langle L \circ J(\varepsilon_{\hat{d}}), \varepsilon_{\hat{a}}\rangle - \\
&\quad - \langle L \circ J^3(\varepsilon_{\hat{d}}), \varepsilon_b\rangle\langle J(\varepsilon_c), \varepsilon_{\hat{a}}\rangle + \langle L \circ J^3(\varepsilon_c), \varepsilon_b\rangle\langle J(\varepsilon_{\hat{d}}), \varepsilon_{\hat{a}}\rangle + \\
&\quad +2\langle L \circ J^3(\varepsilon_c), \varepsilon_{\hat{d}}\rangle\langle J(\varepsilon_b), \varepsilon_{\hat{a}}\rangle + 2\langle J^3(\varepsilon_c), \varepsilon_{\hat{d}}\rangle\langle L \circ J(\varepsilon_b), \varepsilon_{\hat{a}}\rangle \\
&= R_{\hat{a}bc\hat{d}} + L_c^h\,\delta_h^a\,\delta_b^d + L_b^h\,\delta_h^d\,\delta_c^a + L_c^h\,\delta_h^a\,\delta_b^d + \\
&\quad L_b^h\,\delta_h^d\,\delta_c^a + 2L_c^h\,\delta_h^d\,\delta_b^a + 2L_b^h\,\delta_h^a\,\delta_c^d \\
&= R_{\hat{a}bc\hat{d}} + L_c^a\,\delta_b^d + L_b^d\,\delta_c^a + L_c^a\,\delta_b^d + L_b^d\,\delta_c^a + 2L_c^d\,\delta_b^a + 2L_b^a\,\delta_c^d \\
&= R_{\hat{a}bc\hat{d}} + 2(L_c^a\,\delta_b^d + L_b^d\,\delta_c^a + L_c^d\,\delta_b^a + L_b^a\,\delta_c^d).
\end{aligned}$$

Analogously,

$$\begin{aligned}
B_{\hat{a}\hat{b}cd} &= R_{\hat{a}\hat{b}cd} + \langle \varepsilon_d, \varepsilon_{\hat{b}}\rangle\langle L(\varepsilon_c), \varepsilon_{\hat{a}}\rangle - \langle \varepsilon_c, \varepsilon_{\hat{b}}\rangle\langle L(\varepsilon_d), \varepsilon_{\hat{a}}\rangle + \\
&\quad + \langle L(\varepsilon_d), \varepsilon_{\hat{b}}\rangle\langle \varepsilon_c, \varepsilon_{\hat{a}}\rangle - \langle L(\varepsilon_c), \varepsilon_{\hat{b}}\rangle\langle \varepsilon_d, \varepsilon_{\hat{a}}\rangle - \\
&\quad - \langle J^3(\varepsilon_d), \varepsilon_{\hat{b}}\rangle\langle L \circ J(\varepsilon_c), \varepsilon_{\hat{a}}\rangle + \langle J^3(\varepsilon_c), \varepsilon_{\hat{b}}\rangle\langle L \circ J(\varepsilon_c), \varepsilon_{\hat{a}}\rangle - \\
&\quad - \langle L \circ J^3(\varepsilon_d), \varepsilon_{\hat{b}}\rangle\langle J(\varepsilon_c), \varepsilon_{\hat{a}}\rangle + \langle L \circ J^3(\varepsilon_c), \varepsilon_{\hat{b}}\rangle\langle J(\varepsilon_d), \varepsilon_{\hat{a}}\rangle + \\
&\quad +2\langle L \circ J^3(\varepsilon_c), \varepsilon_d\rangle\langle \varepsilon_{\hat{b}}), \varepsilon_{\hat{a}}\rangle + 2\langle J^3(\varepsilon_c), \varepsilon_d\rangle\langle L \circ J(\varepsilon_{\hat{b}}), \varepsilon_{\hat{a}}\rangle \\
&= 0 + L_c^h\,\delta_h^a\,\delta_d^b - L_d^h\,\delta_h^a\,\delta_c^b + L_d^h\,\delta_h^b\,\delta_c^a - L_c^h\,\delta_h^b\,\delta_d^a - \delta_d^b\,L_c^h\,\delta_h^a + \\
&\quad + \delta_c^b\,L_d^h\,\delta_h^a - L_d^h\,\delta_h^b\,\delta_c^a + L_c^h\,\delta_h^b\,\delta_d^a \\
&= L_c^a\,\delta_d^b - L_d^a\,\delta_c^b + L_d^b\,\delta_c^a - L_c^b\,\delta_d^a - L_c^a\,\delta_d^b + L_d^a\,\delta_c^b - L_d^b\,\delta_c^a + L_c^b\,\delta_d^a = 0.
\end{aligned}$$

Therefore,

$$B_{\hat{a}bc\hat{d}} = B_{b\hat{a}\hat{d}c} = -B_{\hat{a}b\hat{d}c} = -B_{b\hat{a}c\hat{d}} = A_{bc}^{ad} + 8L_{(b}^{(a}\,\delta_{c)}^{d)}. \tag{91}$$

All other components of the tensor are evidently equal to zero.

Definition 37. A generalized Kaehler manifold is called *Bochner-flat*, or *generalized Kaehler-Bochner manifold*, if its Bochner tensor is equal to zero identically.

Let M be Bochner-flat generalized Kaehler manifold. In view of (91) it is eqivalent to

$$A_{bc}^{ad} = -8L_{(b}^{(a}\,\delta_{c)}^{d)}, \qquad \text{or} \qquad A_{bc}^{ad} = t_{(b}^{(a}\,\delta_{c)}^{d)},$$

where $t_b^a = -8L_b^a$. In view of Theorem 43 we get the following result:

Theorem 45. *Four-dimensional generalized Kaehler manifold is self-dual iff it is Bochner-flat.* □

6.4 4-dimensional anti-self-dual $\mathcal{K}_\alpha$-manifolds

Let M be an arbitrary four-dimensional anti-self-dual generalized Kaehler manifold, $\vartheta \in \Lambda^+(M)$. Then $W(\vartheta) = 0$, and by (77) and (78) we have:

1) $W(\vartheta)^b{}_a = B^{bd}_{ac}\,\vartheta^c_d = 0$. In view of Lemma 4, the equality will be written in the form $(B^{b0}_{a0} + B^{b1}_{a1})iy = 0$, and in view of arbitrary choice of $y \in C^\infty(M)$, $B^{bc}_{ac} = 0$, i.e. $A^{bc}_{ac} + r^b_a - \frac{1}{6}s\,\delta^b_a = 0$, and thus, $\frac{1}{6}s\,\delta^b_a = 0$, i.e. $s = 0$. Inversely, if $s = 0$, then $B^{bc}_{ac} = 0$, and thus, $W(\vartheta)^b{}_a = 0$.

2) $W(\vartheta)_{ab} = W_{ab}{}^{cd}\,\vartheta_{cd} = 0$. In view of Lemma 4, the equality will be rewritten in the form $W_{ab}{}^{01}(x + iz) = 0$, and in view of arbitrary choice of $x, z \in C^\infty(M)$, $W_{ab}{}^{01} = 0$, and thus, $W_{ab}{}^{cd} = 0$. By (77) the above is equivalent to

$$r^c_a\,\delta^d_b + r^d_b\,\delta^c_a - r^c_b\,\delta^d_a - r^d_a\,\delta^c_b = \frac{1}{3}\,\delta^{cd}_{ab}. \tag{92}$$

Contracting it by indices b and d, we get: $2r^c_a + \frac{1}{2}s\,\delta^c_a - 2r^c_a = \frac{1}{3}s\,\delta^c_a$, hence, $s = 0$. Inversely, if $s = 0$, then (92) will be written in the form $r^c_a\,\delta^d_b + r^d_b\,\delta^c_a - r^c_b\,\delta^d_a - r^d_a\,\delta^c_b = 0$, that is equivalent to $r^0_0 + r^1_1 = 0$, i.e. $s = 0$, and thus, $W(\vartheta)_{ab} = 0$. Hence,

$$W(\vartheta) = 0 \iff s = 0,$$

and we get the following result:

Theorem 46. *A 4-dimensional generalized Kaehler manifold is anti-self-dual iff it is manifold of zero scalar curvature.* □

Remark. If M is a 4-dimensional compact regular spinor manifold [7] carrying a Kaehler structure of classic type, this result can be essentially strengthened. Namely, in this case signature $\tau(M)$ of manifold M is computed by the formula [5]:

$$\tau(M) = \frac{1}{12\pi^2}\int\limits_M (\|W^+\|^2 - \|W^-\|^2)\eta_g.$$

If M is anti-self-dual and not conformal-flat, then it follows that $\tau(M) < 0$. But then $\hat{A}$-kind of the manifold [5], in 4-dimensional case equal to $\frac{1}{16}\tau(M)$, is also negative, and by the Theorem of A.Lichneriwicz [21] M does not admit metrics of positive scalar curvature. By the Theorem of Kazdan-Warner [22], the initial metric on M, being by Theorem 46 a metric of zero scalar curvature, is Ricci-flat. If M is conformal-flat, then since it is a regular Kaehler surface, its first Betti number is zero, i.e. the universal covering space is compact and, by Theorem 46, it has zero scalar curvature, but it is impossible in view of Theorem 44.

Thus, in this case M is Ricci-flat. Inversely, if M is Ricci-flat, then by Theorem 46 it is anti-self-dual. Thus, we get the following result:

Theorem 47. *A 4-dimensional compact regular spinor manifold carrying any Kaehler structure of classical type is anti-self-dual iff it is Ricci-flat.* □

An example of such manifold is a K_3-*surface*, i.e. a compact regular surface with zero first Chern class. It is known [23] that such surfaces form a special type in Kodaira classification of complex surfaces and they are well studied. It

is known, in particular, that all K_3-surfaces are diffeomorphic and their Kaehler structures induced by Calabi-Yau metric are Kaehler-Einsteinian structures [5]. Moreover, by the known result of Hitchin [8] any compact anti-self-dual Einsteinian manifold is either flat or covered by a K_3-surface, having a Calabi-Yau metric.

References

[1] M. Berger, *Remarques sur le Groupe d'Holonomie des Variétés Riemanniennes*, C. R. Acad. Sci. Paris **262** (1966), 316-318.

[2] S.M. Salamon, *Quaternionic Kaehler Manifolds*, Invent. Math. **67** (1982), 143-171.

[3] M.F. Atiyah, N.J. Hitchin, I.M. Singer, *Self-duality in four-dimensional Riemannian Geometry*, Proc. Roy. London **A362** (1978), 425-461.

[4] S. Sternberg, *Lectures on Differential Geometry* (1970), Mir, Moscow (Russian).

[5] A. Besse, *Einsteinian Manifolds*, **I-II** (1990), Mir, Moscow, (Russian).

[6] M. Berger, *Sur le Groupes d'Holonomie des Variétés à Connexion Affine et des Variétés Riemanniennes*, Bull. Soc. Math. France **83** (1955), 279-330.

[7] R. Penrose, *The Twistor Programme*, Math. Phys. Repts. **12** (1977), 65-76.

[8] N.J. Hitchin *On Compact Four-dimensional Einstein manifolds*, J. Diff. Geom. **9** (1974), 435-442.

[9] N.J. Hitchin, *Kahlerian Twistor Spaces*, Proc. London Math. Soc. **43** (1981), 133-150.

[10] B.-Y. Chen, *Some Topological Obstructions to Bochner-Kaehler Metrics and their Applications* J. Diff. Geom. **13** (1978), 547-558.

[11] J.-P. Bourguignon, *Les Variétés de Dimension 4 a Signature non Nulle dont la Courbure est Harmonique sont d'Einstein*, Invent. Math. **63** (1981), 263-286.

[12] A. Derdzinski, *Self-dual Kahler Manifolds and Einstein Manifolds of Dimensional Four*, Comp. Math. **19** (1983), 405-433.

[13] M. Itoh, *Self-duality of Kähler Surfaces*, Comp. Math. **51** (1984), 265-273.

[14] V.F. Kirichenko, *Generalized Hermitian Geometry Methods in Theory of Almost Contact Manifolds*, in book Itogi Nauki i Tehniki: Problemy Geometrii **18** (1986), 25-72 (Russian).

[15] P. Libermann, *Sur le Problèmé d'Equivalence de Certaines Structure Infinitesimales*, Ann di Mathematica **36** (1951), 247-261.

[16] A.P. Shirokov, *On One Space Class over Algebras*, Izv. VUZov. Matem. **1** (1961), 163-170 (Russian).

[17] V.F. Kirichenko, *Axioms of Holomorphic Planes in Generalized Hermitian Geometry*, Dokl. Akad. Nauk SSSR **260** (1981), 795-799 (Russian).

[18] A. Derdzinski, *Examples of Kaehler-Einstein Self-dual Metrics on Complex Plane*,in book Four-dimensional Riemannian Geometry. Arthur Besse Seminar 1978-1979. Mir, Moscow (1985), 280-289 (Russian).

[19] S. Bochner, *Curvature and Betti Numbers, II* Ann. Math. **50** (1949), 77-93.

[20] S. Tachibana, *On Bochner Curvature Tensor*, Not. Sci. Ochanomidzu Univ. **18** (1967), 15-19.

[21] A. Lichnerowicz, *Spineur Harmoniques*, C. R. Acad. Sci. Paris **257** (1963), 7-9.

[22] J.L. Kazdan, F.W. Warner, *A Direct Approach to the Determination of Gaussian and Scalar Curvature Function*, Invent. Math. **28** (1975), 227-230.

[23] K. Kodaira, *On the Structure of Compact Complex Analytic Surfaces, I* Amer. J. Math. **86** (1964), 751-798.